

Nonholonomic Clifford and Finsler Structures,  
Non-Commutative Ricci Flows,  
and Mathematical Relativity

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**Habilitation Thesis**

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**key directions:** mathematical physics, geometric methods in physics,  
general relativity and modified gravity theories, applied mathematics

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## 0.1 Summary of the Habilitation Thesis

It is outlined applicant's 18 years research and pluralistic pedagogical activity on mathematical physics, geometric methods in particle physics and gravity, modifications and applications (after defending his PhD thesis in 1994). Ten most relevant publications are structured conventionally into three "strategic directions": 1) **nonholonomic geometric flows evolutions and exact solutions** for Ricci solitons and field equations in (modified) gravity theories; 2) **geometric methods in quantization** of models with nonlinear dynamics and anisotropic field interactions; 3) **(non) commutative geometry, almost Kähler and Clifford structures, Dirac operators and effective Lagrange–Hamilton and Riemann–Finsler spaces.**

The applicant was involved in more than 15 high level multi-disciplinary international and national research programs, NATO and UNESCO, and visiting/sabatical professor fellowships and grants in USA, Germany, Canada, Spain, Portugal, Romania etc. He got support from organizers for more than 100 short visits with lectures and talks at International Conferences and Seminars.

Both in relation to above mentioned strategic directions 1)–3) and in "extension", he contributed with almost 60 scientific works published and cited in high influence score journals (individually, almost 50 %, and in collaboration with senior, 30 %, and young, 20 %, researchers). Applicant's papers are devoted to various subjects (by 15 main directions) in noncommutative geometry and gravity theories; deformation, A-brane, gauge like and covariant anisotropic quantization; strings and brane physics; geometry of curved flows and associated solitonic hierarchies with hidden symmetries; noncommutative, quantum and/or supersymmetric generalizations of Finsler and Lagrange–Hamilton geometry and gravity; algebroids, gerbes, spinors and Clifford and almost Kähler structures; fractional calculus, differential geometry and physics; off-diagonal exact solutions for Einstein - Yang - Mills - Higgs - Dirac systems; geometric mechanics, nonlinear evolution and diffusion processes, kinetics and thermodynamics; locally anisotropic black holes/ellipsoids / wormholes and cosmological solutions in Einstein and modified gravity theories; applications of above listed results and methods in modern cosmology and astrophysics and developments in standard particle physics and/or modified gravity.

Beginning June 2009, the applicant holds time limited senior research positions (CS 1) at the University Alexandru Ioan Cuza (UAIC) at Iași University, Romania. With such affiliations, he published by 25 articles in top ISI journals; more than half of such papers won the so-called "red,

yellow and blue" excellence, respectively, (4,4, 6), in the competition of articles by Romanian authors. During 2009-2011, he communicated his results at almost 30 International Conference and Seminars having support from hosts in UK, Germany, France, Italy, Spain, Belgium, Norway, Turkey and Romania.

For the Commission of Mathematics for habilitation of university professors and senior researchers of grade 1, it is computed [for relevant publications in absolute high influence score journals] this conventional "eligibility triple": (points for all articles; articles last 7 years; number of citations) = (55.9; 28.74; 53) which is higher than the minimal standards (5; 2.5; 12). Taking into account the multi-disciplinary character of research on mathematical physics, there are provided similar data for the Commission of Physics: (59.19; 31.6; 61) which is also higher than the corresponding minimal standards (5; 5; 40).

Future research and pedagogical perspectives are positively related to the fact that the applicant won recently a three years Grant IDEI, PN-II-ID-PCE-2011-3-0256. This allows him to organize a computer-macros basis for research and studies on mathematical and computational physics and supervise a team of senior and young researches on "nonlinear dynamics and gravity".

## 0.2 Sinteza tezei de abilitare (in Romanian)

Este trecută în revistă activitatea de 18 ani de cercetare și didactică prin cumul a applicantului în domenii legate de fizica matematică, metode geometrice în fizica particulelor și gravitație, modificări și aplicații (după susținerea tezei de doctorat în 1994). Zece cele mai relevante publicații sunt structurate convențional în trei "direcții strategice": 1) evoluții neolomone "geometric flows" și soluții exacte pentru solitoni Ricci și ecuații de câmp în teorii de gravitație (modificate); 2) metode geometrice în cuantificarea modelelor cu dinamică neliniară și interacțiuni anisotrope de câmp; 3) geometrie (ne) comutativă, structuri aproape Kähler și Clifford, operatori Dirac și spații efective Lagrange–Hamilton și Riemann–Finsler.

Aplicantul a fost implicat în peste 15 programe internaționale și naționale de cercetare de nivel înalt, multi-disciplinare, OTAN și UNESCO, și granturi pentru profesor în vizită sau sabatic în SUA, Germania, Canada, Spania, Portugalia, România etc. A obținut suport de la organizatori pentru peste 100 vizite scurte cu lecții și comunicări la conferințe și seminare internaționale.

Cu privire la direcțiile 1) – 3) menționate mai sus, cât și în extensio, applicantul a contribuit cu circa 60 lucrări științifice publicate și citate în reviste cu punctaj înalt de influență (individual, circa 50 %, și în colaborare cu cercetători seniori, 30 %, și tineri, 20 %). Lucrările applicantului sunt consacrate diferitor subiecte (circa 15 direcții principale) în geometrie neocomutativă și teorii de gravitație; cuantificare de deformare, A-brane, similar gauge și covariant anisotropă; fizică string și brane; geometrii "curved flows" și ierarhii solitonice asociate cu simetrii ascunse; generalizări neocomutative, cuantice și/ sau supersimetrice ale geometriilor Finsler și Lagrange–Hamilton și gravitație; algebroizi, gerbe, spinori și structuri aproape Kähler; calculul fracțional, geometrie diferențială și fizică; soluții ne-diagonale exacte pentru sisteme Einstein - Yang - Mills - Higgs - Dirac; geometrie mecanică, evoluție neliniară și procese de difuzie; goluri negre / elipsoizi / wormholuri și soluții cosmologice în teorii de gravitație Einstein și modificate; aplicații ale rezultatelor menționate mai sus în cosmologie modernă și astrofizică și dezvoltări în fizica particulelor standardă și / sau gravitație modificată.

Începând cu iunie 2009, applicantul are poziții de cercetător științific superior, CS 1, cu termen limitat, la Universitatea Alexandru Ioan Cuza (UAIC) din Iași, România. Având astfel afiliere, a publicat peste 25 articole în reviste top ISI; peste o jumătate din articole au fost câștigătoare de tipul "roșu, galben și albastru", respectiv, (4,4,6), în competiția de lucrări ale autorilor care activează în România. În decursul 2009–2011, el a comunicat

rezultatele sale la circa 30 conferințe și seminare internaționale avînd suport de la gazde științifice în MB, Germania, Franța, Italia, Spania, Belgia, Norvegia, Turcia și România.

Pentru comisia de matematică pentru abilitarea profesorilor universitari și a cercetătorilor superiori de gradul 1 este calculat [publicații relevante în reviste cu scor înalt de influență] acest convențional "triplu de eligibilitate": (puncte pentru toate articolele; articole în ultimii 7 ani; numărul citărilor)  $= (55.9; 28.74; 53)$  ce depășește standardele minimale (5; 2.5; 12). Luând în considerație caracterul multi-disciplinar al cercetării din fizică matematică, sunt prezentate date similare pentru comisia de fizică: (59.19; 31.6; 61), ce tot depășesc standardele minimale (5; 5; 40).

Perspective reale de cercetare și activitate pedagogică pentru viitor sunt legate de faptul că applicantul a câștigat recent, pentru trei ani, un Grant IDEI, PN-II-ID-PCE-2011-3-0256. Aceasta îi permite să organizeze o bază computer-macros pentru cercetare și studii în matematică și fizică computațională și conducere a unei echipe de cercetători seniori și tineri în "dinamică nelineară și gravitație".

# Chapter 1

## Achievements

Today, various directions in modern geometry and physics are so interrelated and complex that it is often very difficult to master them as separated subjects. Research and pedagogical activities on mathematical physics, geometry and physics, relativity and high energy physics etc play a multi- and/or inter-disciplinary character with various applications and connections to advanced computer methods and graphics, modern technology and engineering etc. There is a need of research teams of mathematicians skilled both in geometric and analytic methods and oriented to fundamental and experimental physics and/or, inversely, theoretical and mathematical physics researches with a rigorous education and research experience in differential geometry, nonlinear analysis, differential equations and computer methods.

The author of this Habilitation Thesis was involved in various multi-disciplinary research projects and pluralistic pedagogical activity on mathematics and physics<sup>1</sup> after he got his PhD on theoretical physics, in 1994, at the University Alexandru Ioan Cuza (UAIC) at Iași, Romania. The PhD thesis was elaborated almost individually at the Department of Physics of "M. Lomonosov" State University (Moscow, Russia) and Institute of Applied Physics, Academy of Sciences of Moldova (Chișinău, Republic Moldova) during 1984-1992,<sup>2</sup> and finalized at UAIC (1992-1994). That research on geo-

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<sup>1</sup>see Chapter 3 with a relevant Bibliography and CV and publication list (included in the File for this Habilitation Thesis) and, for instance, reviews in MathSciNet and Web of Science and Webpage <http://www.scribd.com/people/view/1455460-sergiu> with details on NATO and UNESCO and visiting/sabatical professor fellowships and grants for various long/short terms visits (respectively, almost 15/ 40) and research activity in USA, UK, Germany, France, Italy, Spain, Portugal, Greece, Norway, Canada, Turkey, Romania etc

<sup>2</sup>in former URSS, there were some options for performing/defending equivalents of PhD and Habilitation Thesis as an individual applicant

metric and twistor methods in classical field theory, gravity and condensed matter physics was developed and extended to new directions in modern geometry, mathematics and physics which are summarized and concluded in this Chapter (see section 1.1 on various directions of research, related papers and comments).

The goal of Chapter 1 is to present applicant's research, professional and academic achievements in relevant (multi/ inter-) disciplinary directions providing necessary proofs and references.<sup>3</sup> His original results are emphasized in a context of present International and National matter of state of science and education. It is used a selection of works and monographs from S. Vacaru's Publication List, see Refs. [1]–[91], and a list of his last 7 years participations/talks at International Conferences and Seminars, see Refs. [92]–[131] (see details in Chapter 3). Taking into account the length limits for such a thesis, there are included in the Bibliography only a part of applicant's publications and recent talks; necessary references and comments on "other" author's papers, and "other" authors, can be found in the cited works.

Section 1.1 is devoted to scientific visibility and prestige of applicant's activity. There are briefly outlined and commented the strategic and main research results and relevant author's references, listed a series of examples and contributions for International Scientific Conferences and Seminars and mentioned most important grants and temporary positions. Comments and conclusions on "main stream and other" important issues and publications and an analysis of eligibility and minimal standards are provided. Section 1.2 contains a review for experts on differential geometry, mathematical physics and gravity theories based on a selection of results from 10 most relevant and important author's articles [1]–[10]. The main goal of this section is to show some most important examples of original research with an advanced level of mathematical methods and possible applications in physics and geometric mechanics which can be found in applicant's works.

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<sup>3</sup>This review type work is performed following CNATDCU Guide on Habilitation Theses, see link <http://www.cnatdcu.ro/wp-content/uploads/2011/11/Ghid-de-abilitare-2012.pdf> . In order to facilitate readers and/or experts from Countries with possible different standards on habilitation, we summarize in English some most important requests stated there in Romanian. Following points (a) (i) in the Guide for such a thesis, the limits are between 150.000–300.000 characters for a Chapter based on most important contributions and selected (maximum 10) most important and relevant author's articles [1]–[10] attached to the "Habilitation File" (with necessary documents presented to the Commission).



## 1.1 Scientific Visibility & Prestige

Conventionally, applicant's research activity correlated to 10 most relevant works [1]–[10] (see a survey in Section 1.2) can be structured into **three strategic directions**:

1. Nonholonomic commutative and noncommutative geometric flows evolutions and exact solutions for Ricci solitons and field equations in (modified) gravity theories [4, 6, 3, 9];
2. Geometric methods in quantization of models with nonlinear dynamics and anisotropic field interactions [7, 5, 10, 9];
3. (Non) commutative geometry, almost Kähler and Clifford structures, Dirac operators, effective Lagrange–Hamilton and Riemann–Finsler spaces and analogous/ modified gravity [2, 1, 8, 6].

In a more general context, including other partner works (inter-related), one can be considered **15 main research directions**.

### 1.1.1 Comments on strategic and main directions

It is presented a synopsis of related ISI works [1]–[64].<sup>4</sup> There are outlined motivations, original ideas and most important results in 15 main directions.

1. *(Non) commutative gauge theories of gravity, anisotropic generalizations, and perturbative methods of quantization* [2, 30, 11, 8, 28, 28, 87, 70, 71, 72, 77, 78, 49, 53].
  - (a) Affine and de Sitter models of gauge gravity.
  - (b) Gauge like models of Lagrange–Finsler gravity.
  - (c) Locally anisotropic gauge theories and perturbative quantization.
  - (d) Noncommutative gauge gravity.

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<sup>4</sup>One should be mentioned here some series of "not less" important contributions containing various preliminary, or alternative, ideas and results published in Romania [65]–[69] and Republic of Moldova [76]–[80]; monographs [70]–[72], chapters and sections in collections of works [81]–[83], reviews in journals and encyclopedia [73, 74], articles in Proceedings of Conferences [84]–[91] and some recent electronic preprints with reviews, computation details and proofs, see [132, 133].

*Comments:* This direction was elaborated as a natural development of some chapters and sections in author's PhD thesis (1994), where affine and de Sitter gauge like models were considered for the twistor–gauge formulation of gravity. The first publications on anisotropic gauge gravity theories were in R. Moldova [77, 78] (1994-1996); see also a paper together with a graduate student, Yu. Goncharenko [11], when authors were allowed to present their results in a Western Journal. The main constructions were based on the idea that the Einstein equations can be equivalently reformulated as some Yang–Mills equations for the affine and/or de Sitter frame bundles, with nonlinear realizations of corresponding gauge groups and well defined projections on base spacetime manifolds (we used the Popov–Dikhhin approach, 1976, and A. Tseytlin generalization, 1982; see references in above cited papers<sup>5</sup>).

In order to formulate (non) commutative and/or supersymmetric gauge theories of Lagrange–Finsler gravity, we used the Cartan connection in the affine and/or de Sitter bundles on Finsler (super) manifolds and various anisotropic generalizations, including higher order tangent/vector bundles. Such results are contained in some chapters of monographs [70, 71, 72] and presented at a NATO workshop in 2001, see [87].

Formal re–definitions of Einstein gravity and generalizations as gauge like models allowed the applicant to perform one of the most cited his works [2] (included as the second one in the list of most relevant applicant's 10 articles). That paper was devoted to the Seiberg–Witten transforms and noncommutative generalizations of Einstein and gauge gravity. The corresponding gravitational equations with noncommutative deformations can be integrated in very general off–diagonal forms [30], see Ref. [8] on noncommutative Finsler black hole solutions.

It should be mentioned here a collaboration with Prof. H. Dehnen (Konstanz University, Germany, 2000-2003) on higher order Finsler–gauge theories, nearly autoparallel maps and conservation laws, see [27, 28] and a recent approach to two–connection perturbative quantization of gauge gravity models [49, 53].

2. *Clifford structures and spinors on nonholonomic manifolds and generalized Lagrange-Finsler and Hamilton-Cartan spaces* [1, 71, 70, 33,

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<sup>5</sup>in this thesis with explicit limits on length, there are provided references only on applicant's works; contributions by other authors are cited exactly in the mentioned references and/or 10 most relevant articles

38, 91, 51, 80, 15, 14, 88, 83, 20, 22, 65, 66, 36, 69, 31, 6].

- (a) Definition of spinors and Dirac operators on generalized Finsler spaces.
- (b) Clifford structures with nonlinear connections and nonholonomic manifolds.
- (c) Spinors and field interactions in higher order anisotropic spaces.
- (d) Solutions for nonholonomic Einstein–Dirac systems and extra dimension gravity.
- (e) Nonholonomic gerbes, index theorems, and Clifford–Finsler geometry.
- (f) Nonholonomic Clifford and Lagrange–Finsler algebroids.

*Comments:* A nonholonomic manifold/bundle space is by definition enabled with a nonholonomic (equivalently, anholonomic, or non-integrable) distribution, see main concepts and definitions in "preliminaries" of the section 1.2.1 and references therein. For various important geometric and physical models, it is enough to consider spaces with nonholonomic splitting (as a Whitney sum) into conventional horizontal (h) and vertical (v) subspaces.<sup>6</sup> One could be conceptual and technical difficulties in adapting the geometric and physical constructions on certain spaces enabled with N-connection structure. For instance, the problem of definition of spinors and Dirac operators on nonholonomic manifolds and/or Finsler–Lagrange spaces was not solved during almost 60 years after first E. Cartan's monographs on spinors in curved spaces and Finsler geometry (during 1932–1935). The applicant proposed rigorous geometric definitions of Finsler spinors [1] and, in general, of spinors and Dirac operators on nonholonomic manifolds/bundle spaces [15, 14, 70, 71, 72], and developed the so-called nonholonomic Clifford geometry in a numbers of his and co-author works during 1995 – present.

There were some attempts to define two dimensional spinor bundles on Finsler spaces and generalizations in the 70th-80th of previous century (Takano and Ono, in Japan, and Stavrinou, in Greece; see main references and historical remarks in [1, 71, 70, 33]). Nevertheless, there were

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<sup>6</sup>Typical examples are 2+2 frame decompositions in general relativity and vector/tangent bundles enabled with nonlinear connection structure (in brief, N-connection, which can be defined as a non-integrable h-v-splitting of the tangent bundle to a manifold, or to a tangent/vector bundle), for instance, in a model of Finsler geometry.

not provided in those works any self-consistent definitions of spinors and Dirac operators for Finsler spaces which would relate a Clifford algebra structure, and spin operators, to Finsler metrics and connections. The problem of definition of "Finsler spinors" is very important in fundamental physics and mechanics if there are considered dependencies of physical objects on "velocity/momentum" variables. For instance, such models of Finsler spacetimes are elaborated for quantum gravity and modern cosmology, see details and critical remarks in [38, 91, 51] and Introduction to [70]. Without spinors/fermions, it is not clear how to construct "viable" physical models with dependence on some "velocity/momentum" type variables. Similar problems have to be solved for generic off-diagonal solutions in Einstein gravity with spinors, and modifications, and nontrivial N-connection structure and conventional spacetime splitting.

In 1994–1995, the applicant became interested in the problem of elaborating theories of gravitational and matter field interactions on generalized Finsler spaces (in a more general context, in the sense of G. Vrănceanu's definition of nonholonomic manifolds, 1926-1927). It was a special research grant Romania–R. Moldova affiliated to the school on generalized Finsler-Lagrange-Hamilton geometry at Iași supervised by Acad. R. Miron. The paper [80] (submitted in 1994 before establishing that collaboration and published in 1996 in R. Moldova) contains the first self-consistent definition of Clifford structures and spinors for Finsler spaces and generalizations. Such results formulated in a more rigorous form, with developments for complex and real spinor Lagrange–Finsler structures and Dirac operators adapted to N-connections, were published also in J. Math. Physics. (1996), see [1].

Having defined nonolonomic Clifford bundles, it was possible to construct geometric models of gravitational and field interactions on (super) spaces with higher order anisotropy [15, 14]. There were obtained some new results in differential spinor geometry and supergeometry with possible applications in high energy physics (see more details in point 4b). It was possible to involve in such activities two professors from Greece (P. Stavrinou and G. Tsagas, see monographs [71, 70] and paper [88]) and some young researchers<sup>7</sup>. Together with some sections

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<sup>7</sup>at that time under-graduate and post-graduate students in R. Moldova (N. Vicol and I. Chiosa, see papers [33, 88, 83]) and Romania (F. C. Popa and O. Țințăreanu-Mircea, see [20, 22, 65, 66])

in monograph [72], such works contain a series of new results on Dirac spinor waves and solitons, spinning particles, in Taub NUT anisotropic spaces, solutions for Einstein–Dirac systems in nonholonomic higher dimension gravity, supergravity and Finsler modifications of gravity.

There were elaborated three another directions related to nonholonomic (Finsler) spinors and Dirac operators: For instance, papers [36, 69] (the first one, in collaboration with J. F. Gonzalez–Hernandez, in 2005, a student from Madrid, Spain) are devoted to nonholonomic gerbes, Clifford–Finsler structures and index theorems. Article [31] contains definitions and examples of nonholonomic Clifford and Finsler–Clifford algebroids with theorems on main properties of indices of connections in such spaces.

Finally, in this point, it should be noted that the constructions for the nonholonomic Dirac operators were applied for definition of noncommutative Finsler spaces and Ricci flows in A. Connes sense, see details in Refs. [6] (the 6th most relevant and important applicant’s paper) and in Part III of monograph [70] (there are connections to points 4b and 14e).

3. *Nearly autoparallel maps, nonlinear connections, twistors and conservation laws in Lagrange and Finsler spaces* [12, 82, 81, 27, 28, 72].

*Comments:* The geometry of nearly autoparallel maps (various examples were studied by H. Weyl, A. Z. Petrov and summarized in a monograph published in Russian by N. Sinyukov in 1979) generalizes various models with geodesic and conformal transforms. Some chapters of applicant’s PhD thesis were devoted to such transforms and definition of corresponding invariants and conservation laws for spaces with nontrivial torsion, endowed with spinor/twistor structure etc.

This is an open direction for further research. For instance, the geometry of twistors for curved spaces was studied in Ref. [12] using nearly autoparallel maps. Local twistors were defined on conformally flat spaces and mapped via generalized transforms to more general (pseudo) Riemannian and Einstein spaces. The key result was that even the twistor equations are not integrable on general curved spaces such couples of spinors structures can be defined via nonholonomic deformations and generalize nearly autoparallel maps. Following this approach, we can consider analogs of Thomas invariants and Weyl tensors (in certain generalized forms, with corresponding symmetries and conservation laws). The constructions were generalized for La-

grange and Finsler spaces [82] – it was a collaboration with a former applicant’s student, S. Ostaf.

There are relevant certain results from Refs. [81, 82] (a collaboration with the former PhD supervisor in Romania, Prof. I. Gottlieb) when the A. Moor’s tensor integral was considered, see paper [82]. It was also an article on tensor integration and conservation laws on nonholonomic spaces published by applicant individually in R. Moldova, see [79].

One should be mentioned again the articles [27, 28], in collaboration with Prof. H. Dehnen, where generalized geodesic and conformal maps were considered in (higher order) models of Finsler gravity and in gauge and Einstein gravity.

A part of results in this direction was partially summarized (also in supersymmetric form) in two chapters of monograph [72].

4. *Locally anisotropic gravity in low energy limits of string/ brane theories; geometry of super-Finsler space* [13, 14, 75, 83, 87, 65, 66, 72].
  - (a) Nonholonomic background methods and locally anisotropic string configurations.
  - (b) Supersymmetric generalizations of Lagrange-Finsler spaces.

*Comments:* If some Lagrange-Finsler geometry models are related to real physics, such configurations have to be derived in some low energy limits of (super) string theory. Papers [13, 14] published in very influent score journals, *Annals Phys. (NY)* and *Nucl. Phys. B*; 1997), were devoted to supersymmetric generalizations of theories with local anisotropies and nonholonomic structures (the concept of superspace/superbundle involves a special class of nonholonomic complex distributions). The applicant is the author of Supersymmetry Encyclopedia term ”super-Finsler space” [75]. We note that prof. A. Bejancu introduced nonlinear connections with ”super-fiber” indices in some his preprints at Vest University of Timișoara and in a monograph on Finsler geometry and applications, in 1990. Applicant’s idea was to formulate a rigorous approach to the geometry of N-connections in superspaces via nonolonomic distributions taking as bases of superbundles various classes of supermanifolds.

One of the main problems in such a research on supergeometry and supergravity is that there is not a generally accepted definition of ”supermanifolds” and ”superspaces” - the existing ones differ for global

constructions. Via nonholonomic distributions, the concept of nonlinear connection can be introduced for all considered concepts of superspace which allow to elaborate corresponding models of supersymmetric Lagrange–Finsler geometry. Following the background field method with supersymmetric and  $N$ -adapted derivatives, and a correspondingly adapted variational principle, locally anisotropic configurations can be derived in low energy limits of string theory.

A series of works on supersymmetric models of nonholonomic superspaces and supergravity was elaborated in R. Moldova and Romania and communicated at International Conferences [83, 87, 65, 66] (in collaboration with former applicant's students, N. Vicol, I. Chiosa, and with young researchers from Bucharest-Magurele, F. C. Popa and O. Țîntăreanu-Mircea). Part I of monograph [72] is devoted to the geometry of nonholonomic supermanifolds and possible applications in physics. Here we note that a series of papers on string, brane and quantum gravity were published during last 15 years, in certain alternative ways, by Prof. N. Mavromatos and co-authors from King's College of London.

5. *Anisotropic Taub–NUT spaces and Dirac spin waves and solitonic solutions* [20, 22, 35, 68].

*Comments:* The applicant found a series of applications of his anholonomic deformation method of constructing exact solutions (related to anisotropic generalizations, exact solutions and physical models of Taub–NUT spaces, with Dirac waves, solitons, spinning particles and supersymmetric configurations) after he got some temporary positions at the Institute of Space Sciences, Bucharest–Magurele, Romania, in 2001. It was a collaboration with PhD students F. C. Popa and O. Țîntăreanu-Mircea, see articles [20, 22] published in high influence score journals (Classical and Quantum Gravity and Nuclear Physics B). The direction was latter, in 2006, extended to Ricci flow solutions related to Taub NUT [35, 68] (in collaboration with Prof. M. Vişinescu).

The works cited in this point contain a number of examples of exact solutions constructed in extra dimension and Einstein gravity theories using the  $N$ -connection formalism and nonholonomic frame deformations which originated from Finsler geometry and nonholonomic mechanics. Such results are related to those outlined below in points 8,9 and 11d, 11e.

6. *Nonholonomic anisotropic diffusion, kinetic and thermodynamical processes in gravity and geometric mechanics* [16, 17, 76, 84, 72, 4, 6, 46].
  - (a) Stochastic processes, diffusion and thermodynamics on nonholonomic curved spaces (super) bundles.
  - (b) Locally anisotropic kinetic processes and thermodynamics in curved spaces.

*Comments:* A program of research with applications of Finsler metrics and stochastic processes in biophysics was performed in the 90th of previous century by professors P. Antonelli and T. Zastawniak in Canada. In order to study diffusion processes on locally anisotropic spaces, it was important to define Laplace operators for Finsler spaces (such constructions were proposed by Prof. M. Anastasiei by 1992–1994, who sheared certain information with, at that time a young researcher, S. Vacaru). That researcher, and present applicant, proposed his definitions of Laplace operator using the canonical distinguished connection and the Cartan distinguished connection and corresponding Itô and Stratonovich types of anisotropic stochastic calculus on generalized Finsler space during Iași Academic days in October 1994. Those results with a study of stochastic and diffusion processes on Finsler–Lagrange spaces and vector bundles enabled with nonlinear connection structure were published latter (1995-1996) in R. Moldova and Proceedings of a Conference in Greece, see Refs. [76, 84]. Independently, similar results were published in parallel by P. Antonelli, T. Zastawniak and D. Hrimiuc with applications in biology and biophysics (1995-2004).

Applicant's research was oriented to exploration of locally anisotropic diffusion processes with possible applications in modern physics and cosmology. In 2001, he was able to publish two his papers in *Annals of Physics (Leipzig)* and *Annals of Physics (New York)* on stochastic processes and anisotropic thermodynamics in general relativity and, respectively, on locally anisotropic kinetic processes and non-equilibrium thermodynamics with some applications in cosmology, see Refs. [16, 17]. The main results in those directions were based on the fact that anisotropic processes with additional nonholonomic constraints, in general, with velocity/momentum variables can be adapted to nonholonomic distributions using metric compatible distinguished connections like in Finsler geometry. Here, it should be mentioned that a Russian physicist, A. A. Vlasov, published in 1966 a book on



”statistical distribution functions” where for the theory of kinetics in curved spaces certain classes of Finsler metrics and connections were considered. Applicant’s idea was to generalize the results for Lagrange and Hamilton geometries and their higher order anisotropic (including supersymmetric) extensions. It was shown that the N– connection formalism and adapted frames play a substantial role in definition of anisotropic nonholonomic stochastic and diffusion processes and similarly in kinetics and geometric thermodynamics of constrained physical systems etc. Such constructions were summarized in two chapters of monograph [72].

Perhaps, there is a perspective direction for future investigations related to above mentioned ”diffusion geometry” and analogous thermodynamics. In papers [4, 6, 46], there are considered generalizations of Grisha Perelman’s entropy and thermodynamical functionals for nonholonomic Ricci flows and Lagrange–Finsler evolutions. In equilibrium, such processes can be described as certain Ricci solitonic systems or effective Einstein spaces with nonholonomic constraints. Various classes of solutions of such evolution and effective field equations can be described by stochastic generating functions. To relate the thermodynamical values for Ricci flows to some analogous diffusion processes and ”standard” kinetic and thermodynamic theory, or to black hole thermodynamic processes, is a difficult mathematical physics problem with less known implications in modern physics.

7. *Differential fractional derivative geometry, gravity and geometric mechanics, and deformation quantization* [132, 92, 55, 56, 57, 58, 60, 61].

*Comments:* This is a very recent direction of applicant’s research papers during 2010–2011. The problem of constructing ”fractional derivatives” was studied in a series of classical works by Leibnitz, Riemann and other prominent mathematicians (fractional derivatives should be not confused with ”fractals” and fractional dimensions). At present, there is an increasing number of publications with applications in modern engineering, economics etc. For instance, there is a well known series of works with fractional derivative diffusion by F. Mainardi (last 30 years) and a self-consistent reformulation of physical theories on flat spaces to fractional derivatives was proposed by V. E. Tarasov (beginning 2005).

The main problems in elaborating geometric and gravitational models with fractional derivatives were related to certain very cumbersome

integro-differential relations present in the Riemann–Liouville integral operators. Such fractional derivatives acting on scalars do not result in zero. In papers [132, 92], there were elaborated models of Ricci flows and gravity theories using the so-called Caputo’s fractional derivative transforming scalar values in zero. Such constructions can be re-defined for the Riemann–Liouville fractional derivatives via corresponding nonholonomic integro-differential transforms.

In a series of works [55, 56, 57, 58], in collaboration with Prof. D. Baleanu (from Ankara, Turkey, and Bucharest–Magurele, Romania), there were elaborated fractional models of almost Kähler – Lagrange geometry, constructed exact solutions in gravity and geometric mechanics, with solitonic hierarchies and deformation quantization of such theories. The results were published in Proceedings of two International Conferences [60, 61] and a seminar in Italy [92].

Finally (in this point), we note that there are not standard and unique ways for constructing geometric and physical models with fractional derivatives. For instance, a series of papers by G. Calcagni (2011) is based on a quite different approach with the aim to unify fractional dimensions, fractional derivatives, noncommutative and diffusion processes. The geometric formalism and related fractional partial derivatives depend on certain assumptions on the types of nonlocal and ”memory” nonlinear effects we try to study, for instance, in theories of condensed matter or quantum gravity.

8. *Geometric methods of constructing generic off-diagonal solutions for Ricci solitons, nonholonomic Einstein spaces and in modified theories of gravity* [4, 49, 50, 133, 85, 90, 30, 52, 54, 40, 41, 42, 70, 63],[17]–[26].
  - (a) Decoupling property of (generalized) Einstein equations and integrability for (modified) theories with commutative and noncommutative variables.
  - (b) Generating exact solutions with ellipsoidal, solitonic and pp-wave configurations, possible cosmological solutions.
  - (c) Generic off-diagonal Einstein–Yang–Mills–Higgs configurations.

*Comments:* The gravitational field equations in Einstein gravity and modifications consist very sophisticated systems of nonlinear partial differential equation (PDE) which can be solved in general form only for some special ansatz (for instance, with diagonal metrics depending on

1-2 variables). A surprising and very important decoupling property of such PDE, and generalizations to geometric flow evolution equations, was found with respect to certain classes of nonholonomic frames with associated N-connection structure. Such frames can be naturally defined, for instance, for a class of nonholonomic splitting 2+2 splitting in general relativity and any 2, or  $3 + 2 + 2+2+\dots$  decomposition with formal fibred structure, up to corresponding frame transforms and deformation of connections, in various modified gravity (with non-commutative, almost Kähler, Finsler type variables etc). In result of such a decoupling, one obtains such sub-systems of PDE which can be integrated, i.e. solved, in very general forms, for various classes of generic off-diagonal metrics (which can not be diagonalized via frame transforms) and generalized connections with nontrivial torsion, see details in section 1.2.6 and Refs. [4, 49, 50, 133]. Imposing additional constraints, we can construct very general classes of solutions for the torsionless and metric compatible Levi-Civita connection.

The idea of general decoupling of gravitational field equations in Einstein, string and Finsler gravity was communicated in 1998 at a conference in Poland [85], see also a more rigorous mathematical approach in [90]. The first examples of different classes of solutions were presented in high influence score journals *Annals of Physics* (NY) and *JHEP* journals, see [17, 18]. A number of new classes and possible physically important off-diagonal solutions with ellipsoid/ toroidal symmetries and/or wormhole, solitons, Dirac waves and nontrivial Einstein-Yang-Mills-Higgs configurations, cosmological solutions etc were studied in Refs. [25, 26, 52, 54].

The so-called anholonomic deformation method of constructing exact solutions in commutative and noncommutative gravity and Ricci evolution theories is perhaps the most general one for "geometric" generating of exact solutions, see a number of additional examples in Refs. [40, 41, 42],[21]–[25],[30, 8, 9]. Parts I and II in collection of works [70] contain both geometric details and examples for solutions in generalize metric-affine and Lagrange-Finsler-affine gravity theories, noncommutative gravity, extra dimension models etc. The possibility to derive off-diagonal solutions with anisotropic scaling, off-diagonal parametric evolution, dependence on generating and integration functions and parameters seem to be very important in elaborating new models of covariant renormalizable theories of quantum gravity [10, 63].

9. *Warped off-diagonal wormhole configurations, flux tubes and propagation of black holes in extra-dimensions* [21, 22, 23, 24, 25, 72].

*Comments:* The geometric methods of constructing solitonic and pp-wave solutions on off-diagonal generalizations of such spacetimes were applied also in a collaboration with Prof. D. Singleton, from California State University at Fresno, USA, and some students from R. Moldova, (2001), see papers [21, 22, 23, 24, 25] and Parts I and II in monograph [72]. This direction of research is related to that outlined above in point 5 and provided explicit examples of application of the methods mentioned in point 8.

Such results were cited in a series of works on brane gravity because the applicant and co-authors were able to provide explicit applications of the anholonomic deformation method of constructing exact solutions with nonlinear off-diagonal warped interactions, non-compactified extra dimensions and locally anisotropic gravitational configurations.

10. *Solitonic gravitational hierarchies in Einstein and Finsler gravity* [47, 45, 18, 20, 23, 24, 32, 56, 72].

*Comments:* It was a collaboration with prof. S. Anco from Brock University, Ontario, Canada, during applicant's visiting professor position in 2006. It was known that the geometry of curve flows on spacetimes with constraint curvature coefficients encode as bi-Hamilton systems various data for solitonic hierarchies and corresponding sine-Gordon, Kadomtzev-Petviashvili and other type solitonic equations. For more general classes of geometries, such a program was considered less realistic because of general dependence of Riemann curvature, Ricci and (possible) tensors on spacetime coordinates.

The applicant used his expertise in generalized Finsler geometry and nonholonomic deformations of geometric structures. The main idea was to construct from a prescribed Finsler fundamental generating function, i.e. metric, via corresponding N-connection splitting and frame transform, following a well defined geometric structure, an auxiliary connection for which the curvature tensor is determined by constant coefficients with respect certain classes of N-adapted frames. In such cases, the geometric data for Finsler geometry (and various generalizations) can be encoded into solitonic hierarchies, see Ref. [45] (together with Prof. S. Anco).

The conventional N-connection splitting can be considered on (pseudo) Riemannian (in particular, Einstein) spaces which also allows us to

redefine equivalently the geometric/physical data in terms of necessary type auxiliary connections. Solitonic hierarchies can be derived similarly as in Lagrange–Finsler geometry but mimicking such structures on nonholonomic (pseudo) Riemann and effective Einstein–Cartan manifolds completely determined by the metric structure, see Ref. [47]. Such an approach provides us with a new scheme of solitonic classification of very general classes of exact solutions in Einstein, Einstein–Finsler and nonholonomic Ricci flow equations.

This direction is related to series of works with solitonic configurations in pp-wave spacetimes, solitonic propagation of black holes in extra dimensions and in modified theories, solitonic wormholes and metric-affine and/or noncommutative models of solitons in gravity and string/brane models, fractional solitonic hierarchies etc, see a number of examples in Refs. [18, 20, 23, 24, 32, 56] and Parts I and II in monograph [72].

11. *Principles of Einstein–Finsler gravity and applications* [38, 51, 91, 72, 78, 11, 39, 70, 71, 8, 9, 63, 64].
  - (a) Classification of Lagrange–Finsler-affine spaces.
  - (b) Critical remarks on Finsler gravity theories.
  - (c) On axiomatics of Einstein–Finsler gravity.
  - (d) Exact solutions in (non) commutative Finsler gravity and applications.
  - (e) (Non) commutative Finsler black holes and branes, black rings, ellipsoids and cosmological solutions.

*Comments:* There were many attempts to develop Finsler generalizations of special and general relativity theories, see reviews of results in Ref. [38] and Introduction to monograph [72]. Here we note certain constructions by Profs. M. Matsumoto and Y. Takano (Japan) and J. Horvath (Hungary) who proposed in the 70-80th of previous century certain analogs of Einstein equations using Finsler connections (for instance, using the Cartan distinguished connection, d-connection). In Romania, such approaches were studied on vector/tangent bundles, including generalized Lagrange spaces, by Acad. R. Miron and Profs. M. Anastasiei, G. Atanasiu, A. Bejancu and others during 1980-1995. There were unsolved, for instance, three very important issues which would prove viability and relation to standard theories of physical

models with Finsler like metrics and N-connection and d-connection structure: 1) to derive exact solutions for Finsler like gravity theories (for instance, what would be some analogs of Finsler black holes, what kind of cosmological solutions can be derived and considered for further research); 2) how to define Finsler spinors; 3) how commutative and noncommutative models of Finsler gravity can be related to string/brane and noncommutative geometry/gravity theories. In points 1-4 above, it is sketched how solutions of such problems were performed in applicant's works, see also references [78, 11] on Finsler – gauge formulations of gravity, with analogous Yang–Mills equations for gravity.

In Part I of monograph [72], an important classification of Finsler spaces and generalizations depending on compatibility of fundamental geometric structures was elaborated. There were considered various classes of metric compatible and noncompatible Finsler d-connections with general nonvanishing torsion structure. Using the anholonomic deformation method (see point 10 above), there were constructed explicit examples of exact solutions in generalized Lagrange-Finsler-affine gravity and analyzed possible physical implications. Here we note Ref. [39] for extensions of nonholonomic gravity and Finsler like theories to nonsymmetric metrics, see also monographs [70, 71] on supersymmetric/spinor and noncommutative Finsler modifications of gravity.

In Ref. [38], it was concluded that most closed to standard theories of physics are the Finsler models with metric compatible d-connections (for instance, the Cartan, or canonical, d-connection) constructed on tangent bundle to Lorentz manifolds. Such theories allows us to define spinor and fermions in form similar to general relativity but on non-holonomic manifolds/bundles. Finsler–Ricci evolution models can be introduced via nonholonomic deformations of the (pseudo) Riemannian ones [64].

Last five years, a series of new Finsler gravity papers (by a number of authors: P. Stavrinos, A. Kouretsis, N. Mavromatos, J. Skakala, F. Girelli, S. Liberati, L. Sindoni, C. Lämmerzahl, V. Perlik, G. W. Gibbons and others) were published in relation to expected Lorentz violations in quantum gravity, anisotropic effects in modern cosmology etc. In a series of papers by Zhe Chang and Xin Li (2009-2010), authors proposed that the Chern d-connection has certain "unique" fundamental properties for generalizations of the Einstein gravity the-

ory. Such constructions were considered to be less adequate for scenarios related to standard physics because of generic nonmetricity in Chern's and Berwald's models of Finsler geometry and gravity, see critical remarks [51].

Applicant's conclusions where that using the canonical d-connection and/or Cartan's d-connection it is possible to construct Einstein – Finsler like theories of gravity on tangent/vector bundles, or on non-holonomic manifolds, following the same principles as in general relativity and the Ehlers–Pirani–Schild (EPS) axiomatics, see references in [91]. Various important issues on modified dispersion relations, Finsler branes and noncommutative black holes, models of quantum gravity etc are considered in Refs. [8, 9, 63].

12. *Stability of nonholonomic gravity and geometric flows with nonsymmetric metrics and generalized connection structures* [39, 42, 43].

*Comments:* The applicant extended his research activity to geometries and physical models with nonsymmetric metrics after two his visits to Perimeter Insitute, Canada (in 2007-2008, hosted by Prof. J. W. Moffat, an expert in such directions, beginning 70th). Such theories were orginally proposed by A. Einstein and L. P. Eisenhardt (1925-1945 and 1951-1952). There were pulished two papers with critical remarks on perspectives in physics for such a direction (by T. Damour, S. Deser and J. McCarthy, 1993, and T. Prokopec and W. Valkenburg, 2006) because of un-physical modes and un-stability of some models. It should be noted that in 1995 an improved model with nonintegrable constants was elaborated by J. Légaré and J. W. Moffat. Nevertheless, questions on stability had to be solved.

The applicant addressed the problem of nonsymmetric metrics in gravity from view point of nonholomic geometric flows characterized by nonsymmetric Ricci tensors [42]. In such cases, under evolution, nonsymmetric components of metrics appear naturally which results also in nonsymmetric Ricci soliton configurations as certain equilibrium states. There were constructed explicit classes of exact solutions with "nonsymmetric" ellipsoids which are stable as deformations of black hole solutions [43]. That was possible by adapting the constructions to certain nonholonomic frames with N-connection structure. In a more general context, such theories can be re-written in almost Kähler and/or Lagrange–Finsler variables [39] which allows us to study various geometric evolution models with symmetric and nonsymmetric

metrics and connections and perform deformation quantization, see next point.

13. *Deformation, A-brane and two-connection and gauge like quantization of almost Kähler models of Einstein gravity and modifications* [7, 37, 34, 67, 29, 47, 5, 44, 49, 53, 55].
  - (a) Almost Kähler and Lagrange–Finsler variables in geometric mechanics and gravity theories.
  - (b) Deformation quantization of generalized Lagrange–Finsler and Hamilton–Cartan theories.
  - (c) Fedosov quantization of Einstein gravity and modifications.
  - (d) A–brane quantization of gravity.
  - (e) Two–connection quantization of Einstein, loops, and gauge gravity theories.

*Comments:* It is of primary importance in modern physics to formulate a viable model of quantum gravity (QG). Various ideas, approaches and techniques were proposed but up till present it is far to say that we could overcome the problems arising in each quantization scheme. Gravity is a generic nonlinear theory; not having a well defined mathematical branch of nonlinear functional analysis, it is not possible to formulate a unique and rigorous scheme; we still have to search for new experimental data and relate the constructions to phenomenological models in modern cosmology and high energy physics.

During last 7 years, the applicant published in high influence score journals a series of papers on geometric methods in quantum gravity: The first direction he addressed was that on deformation (Fedosov) quantization of Lagrange–Finsler and gravity theories with nonholonomic variables [34, 67]. The main idea was to use some very important results (due to A. Karabegov and M. Schlichenmeier, 2001) on deformation quantization (DQ) of almost Kähler geometries. Reformulating Lagrange–Finsler geometries in almost symplectic/ Kähler variables, the scheme of DQ can be naturally extended to various spaces admitting formal such parametrizations. In Refs. [7, 37], the approach was extended to gravitational theories by prescribing a corresponding N–connection structure which allows to define some effective almost Kähler variables. So, the Fedosov method, in nonholonomic variables, can be applied to quantize the Einstein and modified theories in a sense of the DQ paradigm.



A series of results on DQ of Lagrange and Hamilton–Cartan geometries were obtained in collaboration with Prof. F. Etayo and Dr. R. Santamaria (University of Cantabria, Santander, Spain; 2005), see [29]. When the applicant, being at Fields Institute at Toronto (Canada), got also an associated professor position at UAIC he performed a common research with Prof. M. Anastasiei [47]. It should be noted here that for Hamilton configurations on co–tangent bundle, the geometry of phase space posses additional symplectic symmetries which result in a very complex structure of induced N–connections and linear connections. The DQ scheme has to be applied in a quite different form for Lagrange spaces, i.e. on tangent bundles, and for Hamilton spaces, or any other geometries on co–tangent bundles. In the last case, a more advanced geometric techniques adapted to Legendre transforms and almost symplectic structure had to be elaborated. Recently, the DQ formalism was generalized fractional derivative geometries and fractional mechanics and gravity, see [55].

Nevertheless, the DQ scheme is still not considered as a generally accepted procedure with perturbative limits for operators acting on Hilbert spaces etc. For instance, E. Witten and S. Gukov (2007) elaborated an alternative formalism (the so–called brane quantization with A–model complexification). In [5], it was proved that the Einstein gravity in almost Kähler variables can be quantized following the A–model method. Possible connections to other approaches were analyzed in [44] (for loop gravity with Ashtekar–Barbero variables determined by Finsler like connections) and in [49, 53] for the so–called bi–connection formalism and perturbative quantization of gauge gravity models.

14. *Covariant renormalizable anisotropic theories and exact solutions in gravity* [10, 63, 49, 53].
  - (a) Modified dispersions, generalized pseudo–Finsler structures and Hořava–Lifshitz theories on tangent bundles.
  - (b) Covariant renormalizable models for generic off–diagonal space–times and anisotropically modified gravity.

*Comments:* The Newton gravitational constant for four dimensional interactions results in a generic non–renormalizability of the general relativity theory. In the pervious point, we considered various schemes of geometric, non–perturbative and/or gauge like quantization but those constructions do not solve the problem of constructing a viable

model of QG with a perturbative scheme without divergences from the ultraviolet region in momentum space (such methods are requested by phenomenology particle physics and analysis of possible implications in modern cosmology). A recent approach to QG (the so-called Hořava–Lifshitz models, 2009) is developed with nonhomogeneous anisotropic scaling of space and time like variables which allow to develop certain covariant renormalization schemes (in [10], we followed certain ideas due to S. Odintsov, S. Nojiri etc, 2010).

Various models of QG, including those with anisotropic configurations, are with modified dispersion relations which, in their turn, can be associated with certain classes of Finsler fundamental generating functions. In Ref. [63], we developed a formalism for perturbative quantization of such Hořava–Finsler models. In both cases, for constructions from the last two mentioned papers, a crucial role in the quantization procedure is played by the type of nonholonomic constraints, generating functions and parameters which are involved in some families of generic off-diagonal solutions of Einstein equations and generalizations (see point 8 above). In [10, 63] and [49, 53], we proved that the nonlinear gravitational dynamics and corresponding nonholonomic constraints can such way parametrized when certain "renormalizable" configurations survive in an anisotropic form for which a covariant Hořava–Lifshitz quantization formalism can be applied.

15. *Nonholonomic Ricci flows evolution, thermodynamical characteristics in geometric mechanics and (analogous) gravity, and noncommutative geometry* [4, 17, 42, 46, 132, 32, 35, 68, 40, 42].

- (a) Generalization of Perelman's functionals and Hamilton's equations for nonholonomic Ricci flows.
- (b) Analogous statistical and thermodynamic values for evolutions of Lagrange–Finsler geometries and analogous gravity theories.
- (c) Nonholonomic Ricci solitons, exact solutions in gravity, and symmetric and nonsymmetric metrics.
- (d) Geometric evolution of pp-wave and Taub NUT spaces.
- (e) Nonholonomic Dirac operators, distinguished spectral triples and evolution of models of noncommutative geometry and gravity theories.

*Comments:* One of the most remarkable results in modern mathematics, and physics, is the proof of the Poincaré conjecture by Grisha

Perelman (2002-2003) following methods of the theory of Ricci flows (1982). Those constructions were originally considered for evolution of Riemannian and/or Kähler metrics using the Levi–Civita connection.

The applicant became interested in geometric analysis and possible applications in physics beginning 2005 when he was with a sabbatical professor position in Madrid, Spain. His idea was to consider additional nonholonomic constraints on Ricci flows of/on (pseudo) Riemannian and/or vector bundles and study geometric evolution of systems with a more complex geometric structure, as well related modifications of physically important models [4]. Such constructions allow us to study evolution, for instance, of a (pseudo) Riemannian geometry into commutative and noncommutative geometries [6], with symmetric and nonsymmetric metrics and connections [42], Lagrange–Finsler geometries [46], fractional derivative geometric evolution [132]. It is an important task for further research to study subjects related to geometric flows and renormalizations, noncommutative and supersymmetric models of evolution, exact solutions for stationary Ricci soliton configurations and modified gravity theories, possible applications in modern cosmology and astrophysics etc.

In the theory of nonholonomic Ricci flows, the key constructions are related to scenarios of adapting the evolution to N–connection structure in a form preserving certain important geometric/physical values and properties. For instance (in Refs. [32, 35, 68, 40, 42]), there were analyzed various classes of solutions for geometric flows of three and four dimensional Taub NUT spaces, pp–wave and solitonic deformations of the Schwarzschild solution. Such configurations, even in geometric mechanics are characterized by analogous thermodynamics values derived from nonholonomic versions of Perelman’s functionals and associated entropy.

### 1.1.2 Visibility of scientific contributions

Beginning 1994, he published in above mentioned strategic and main 15 directions more than 60 scientific articles in high influence score, and top ISI journals, and three monographs with positive reviews in MathSciNet and/or Zentralblatt, see Refs.[1]–[64] and, additionally, [65]–[91] in Chapter 3. Totally, there are found in arXiv.org and inspirehep.net more than 120 scientific works and preprints with details of computations and alternative ideas and constructions. There are mentioned in Web of Science more than

100 citations (by 60, there are listed in eligibility files attached to this Thesis).

The bulk of most important applicant's publications are in mathematical physics journals: Journal of Mathematical Physics (10 papers), Int. J. Geom. Meth. Mod. Phys. (6 papers), J. Geom. Phys. (2 papers) etc, and theoretical/particle physics journals: Class. Quant. Grav. (5 papers), Nucl. Phys. B (2 papers), JHEP (2 papers), Annals Phys. NY (2 papers), Phys. Lett. A and B (4 papers), Int. J. Theor. Phys. (8 papers) etc.

1. As results of International Competitions the applicant got:
  - three NATO/DAAD senior researcher fellowships for Portugal and Germany, 2001-2004
  - four visiting professor fellowships in Greece, USA and Canada (2001, 2002, 2005-2006)
  - a sabbatical professor fellowship in Spain, 2004-2005
  - a research grant of R. Moldova government, 2000-2001
  - a three years Romanian Government Grant IDEI, PN-II-ID-PCE-2011-3-0256, 2011-2014
2. Two visiting researcher positions related to "scholar at risk status" at Fields and Perimeter Institute, Canada (2006-2008) and other Universities and Research Institutes in different Western Countries; the applicant had a specific research activity derived from his claims of political refugee status from the "communist R. Moldova" during 2001-2009. Here it should be noted some important visits at ICTP, Trieste, Italy (1999), "I. Newton" Mathematical Institute at University of Cambridge, UK (1999) and a recent visit at Albert Einstein Institute, Max Plank Institute, Potsdam, Germany - October, 2010.
3. He got support (in the bulk complete, for travel, accommodations, honorary etc) as an invited lecturer and talks for more than 100 conferences and visits in USA, UK, Germany, Italy, France, Spain, Portugal, Greece, Belgium, Austria, Luxembourg, Norway, Turkey, Poland, Romania etc (certain relevant details are presented in Publication List for the file related to this Habilitation Thesis). We also attach a list of last seven years conferences and typical proceedings at the end of Chapter 3, see respectively [92]–[131] and [81]–[91].

4. *Competitions of Articles:* During 2009-2011, CNCSIS accepted as the best by 14 author's articles with grants about 900 E ("red" 4 articles, [8, 9, 10, 51]) and 450 E ("yellow" 4 articles, [5, 50, 53, 57]) and 110 E ("blue" 6 articles, in 2009, [6, 7, 43, 45, 46, 47]).<sup>8</sup>

Finally, it should be noted that the applicant's mobility was very important and necessary for his research and collaborations.

### 1.1.3 Eligibility, minimal standards and recent activity

The applicant's research activity and main publications can be considered by the Commission of Mathematics (a similar mathematical one evaluated positively applicant's application for a Grant IDEI, in 2011), or by the Commission of Physics, at CNATDCU, Romania. It should be taken into account the multi-disciplinary character of research on mathematical physics. There are a bit different standards for eligibility and evaluation of minimal standards for such Commissions. For instance, it is not allowed to include for consideration by mathematicians the publications in Int. J. Theor. Phys., Rep. Phys. and other journals with less than 0.5 absolute influence score. There are requested at least 12 citations in allowed journals. For physicists, a series of journals with score higher than 0.3 became admissible but there are requested more than 40 citations in an extended class of allowed journals (for experimental and phenomenological physics journals, the number of co-authors the number of publications per year are much higher then similar ones in mathematics and applications and this give rise "statistically" to a grater number of citations).

We note here that for the Commission of Mathematics for habilitation of university professors and senior researchers of grade 1, it might be computed (for publications in relevant "absolute influence score" journals) this conventional "eligibility triple" with corresponding (points for all articles; articles last 7 years; number of citations) = (55.9; 28.74; 53) which is higher than respective minimal standards (5; 2.5; 12) - there are considered 45 published articles. Such details, explanations and calculus are given in the requested evaluation files. As a matter of principle, the applicant became eligible to compete for the most higher positions of university professor/senior researcher CS 1, in Romania, by 1997-1998. Similar data for the Commission of Physics, for 55 articles, can be computed (59.19; 31.6; 61),

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<sup>8</sup>For instance, see the list for 2011, numbers 1250-1252,  
[http://uefscdi.gov.ro/userfiles/file/PREMIERE\\_ARTICOLE/articole%202011/evaluare/REZUTATE%20noiembrie%20ACTUALIZAT%2022%20DECEMBRIE.pdf](http://uefscdi.gov.ro/userfiles/file/PREMIERE_ARTICOLE/articole%202011/evaluare/REZUTATE%20noiembrie%20ACTUALIZAT%2022%20DECEMBRIE.pdf)

which is also higher than the corresponding minimal eligibility standards (5; 5; 40).

All evaluated (and the bulk cited in this thesis) articles got positive reviews in MathSciNet and Zentralblatt (one of them, on nonholonomic Ricci flows, was appreciated in "Nature" being listed in World of Science, Scopus with PDFs dubbed in inspirehep.net and arxiv.org, where a number of citations can be found and checked.<sup>9</sup>

During 2009–2011, with affiliation at University Alexandru Ioan Cuza at Iași, Romania, he published almost 25 top ISI papers on mathematics and physics (more than a half of them being in the "red/yellow/ blue" category for Competition of Articles) and got financial support from organizers for short term visits and invited lectures and talks (more than 30 ones). This would allow the applicant to extend and develop his experience on research and teaching in North America and Western Europe, Romania and former URSS, on supervision PhD and master theses, elaborating monographs and textbooks for university students and delivering lectures and seminars in English, Romanian and Russian.

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<sup>9</sup>The link to Nature Physics, vol. 4., issue 5, pp. 343 (2008) is <http://www.nature.com/nphys/journal/v4/n5/full/nphys948.html#Constant-flow>

For conveniences, it is presented here the text:

*Research Highlights:* Nature Physics 4, 343 (2008), doi:10.1038/nphys948  
J. Math. Phys. 49, 043504 (2008)

Only once, apparently, did Gregorio Ricci-Curbastro publish under the name Ricci. That was in 1900, but the paper — entitled *Methodes de calcul différentiel absolu et leurs applications*, and co-authored with his former student Tullio Levi-Civita — became the pioneering work on the calculus of tensors, a calculus also used by Albert Einstein in his theory of general relativity.

Ricci-Curbastro's short name stuck, and Ricci flow' is the name given to one of the mathematical tools arising from his work. That tool has become known to a wider audience as a central element in Grigori Perelman's proof of the Poincaré conjecture.

Sergiu Vacaru now takes Perelmans work further, going beyond geometrical objects and into the domain of physics with a generalized form of the Ricci-flow theory. In the second paper of a series devoted to these so-called non-holonomic Ricci flows, Vacaru shows how the theory may be applied in tackling physical problems, such as in einsteinian gravity and lagrangian mechanics.

## 1.2 A "Geometric" Survey of Selected Results

The goal of this section is to provide a selection of results from 10 most relevant applicant's publications [1]–[10] containing explicit definitions, theorems and main formulas.<sup>10</sup> Such a brief review is oriented to advanced researchers and experts on mathematical physics and geometric methods in physics.

### 1.2.1 Nonholonomic Ricci evolution

Currently a set of most important and fascinating problems in modern geometry and physics involves the task to find canonical (optimal) metric and connection structures on manifolds, state possible topological configurations and analyze related physical implications. In the past almost three decades, the Ricci flow theory has addressed such issues for Riemannian manifolds. How to formulate and generalize the constructions for non-Riemannian manifolds and physical theories, it is a challenging topic in mathematics and physics. The typical examples come from string/brane gravity containing nontrivial torsion fields and from modern mechanics and field theory geometrized in terms of symplectic and/or generalized Finsler (Lagrange or Hamilton) structures.

The goal of this subsection is to investigate the geometry of evolution equations under non-integrable (equivalently, nonholonomic/ anholonomic) constraints resulting in nonholonomic Riemann–Cartan and generalized Finsler–Lagrange configurations.

#### Preliminaries: nonholonomic manifolds and bundles

A nonholonomic manifold is defined as a pair  $\mathbf{V}=(M, \mathcal{D})$ , where  $M$  is a manifold<sup>11</sup> and  $\mathcal{D}$  is a non-integrable distribution on  $M$ . For certain important geometric and physical cases, one considers N-anholonomic manifolds when the nonholonomic structure of  $\mathbf{V}$  is established by a nonlinear connection (N-connection), equivalently, a Whitney decomposition of the tangent space into conventional horizontal (h) subspace,  $(h\mathbf{V})$ , and vertical (v) sub-

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<sup>10</sup>Cumbersome proofs and references to other authors are omitted. Nevertheless, we shall provide a series of "simplest" examples in order to familiarize readers with such geometric methods. Some "overlap" in denotations and formulas will be possible because they exist in the original published works. In abstract form, this is used for simplifying proofs, for instance, in some models of commutative and noncommutative geometry.

<sup>11</sup>we assume that the geometric/physical spaces are smooth and orientable manifolds

space,  $(v\mathbf{V})$ ,<sup>12</sup>

$$T\mathbf{V} = h\mathbf{V} \oplus v\mathbf{V}. \quad (1.1)$$

Locally, a N-connection  $\mathbf{N}$  is defined by its coefficients  $N_i^a(u)$ ,

$$\mathbf{N} = N_i^a(u)dx^i \otimes \frac{\partial}{\partial y^a}, \quad (1.2)$$

and states a preferred frame (vielbein) structure

$$\mathbf{e}_\nu = \left( \mathbf{e}_i = \frac{\partial}{\partial x^i} - N_i^a(u) \frac{\partial}{\partial y^a}, e_a = \frac{\partial}{\partial y^a} \right), \quad (1.3)$$

and a dual frame (coframe) structure

$$\mathbf{e}^\mu = (e^i = dx^i, \mathbf{e}^a = dy^a + N_i^a(u)dx^i). \quad (1.4)$$

The vielbeins (1.4) satisfy the nonholonomy relations

$$[\mathbf{e}_\alpha, \mathbf{e}_\beta] = \mathbf{e}_\alpha \mathbf{e}_\beta - \mathbf{e}_\beta \mathbf{e}_\alpha = W_{\alpha\beta}^\gamma \mathbf{e}_\gamma \quad (1.5)$$

with (antisymmetric) nontrivial anholonomy coefficients  $W_{ia}^b = \partial_a N_i^b$  and  $W_{ji}^a = \Omega_{ij}^a$ , where  $\Omega_{ij}^a = \mathbf{e}_j(N_i^a) - \mathbf{e}_i(N_j^a)$  are the coefficients of N-connection curvature. The particular holonomic/ integrable case is selected by the integrability conditions  $W_{\alpha\beta}^\gamma = 0$ .

In N-adapted form, the tensor coefficients are defined with respect to tensor products of vielbeins (1.3) and (1.4). They are called respectively distinguished tensors/ vectors /forms, in brief, d-tensors, d-vectors, d-forms.

A distinguished connection (d-connection)  $\mathbf{D}$  on a N-anholonomic manifold  $\mathbf{V}$  is a linear connection conserving under parallelism the Whitney sum (1.1). In local form, a d-connection  $\mathbf{D}$  is given by its coefficients  $\Gamma_{\alpha\beta}^\gamma = (L_{jk}^i, L_{bk}^a, C_{jc}^i, C_{bc}^a)$ , where  ${}^hD = (L_{jk}^i, L_{bk}^a)$  and  ${}^vD = (C_{jc}^i, C_{bc}^a)$  are respectively the covariant h- and v-derivatives.<sup>13</sup>

<sup>12</sup>Usually, we consider a  $(n+m)$ -dimensional manifold  $\mathbf{V}$ , with  $n \geq 2$  and  $m \geq 1$  (equivalently called to be a physical and/or geometric space). In a particular case,  $\mathbf{V} = TM$ , with  $n = m$  (i.e. a tangent bundle), or  $\mathbf{V} = \mathbf{E} = (E, M)$ ,  $\dim M = n$ , is a vector bundle on  $M$ , with total space  $E$ . We suppose that a manifold  $\mathbf{V}$  may be provided with a local fibred structure into conventional "horizontal" and "vertical" directions. The local coordinates on  $\mathbf{V}$  are denoted in the form  $u = (x, y)$ , or  $u^\alpha = (x^i, y^a)$ , where the "horizontal" indices run the values  $i, j, k, \dots = 1, 2, \dots, n$  and the "vertical" indices run the values  $a, b, c, \dots = n+1, n+2, \dots, n+m$ .

<sup>13</sup>We shall use both the coordinate free and local coordinate formulas which is convenient both to introduce compact denotations and sketch some proofs. The left up/lower indices will be considered as labels of geometrical objects. The boldfaced letters will point that the objects (spaces) are adapted (provided) to (with) N-connection structure.



The torsion of a d-connection  $\mathbf{D} = ({}^h D, {}^v D)$ , for any d-vectors  $\mathbf{X} = {}^h \mathbf{X} + {}^v \mathbf{X}$  and  $\mathbf{Y} = {}^h \mathbf{Y} + {}^v \mathbf{Y}$ , is defined by the d-tensor field

$$\mathbf{T}(\mathbf{X}, \mathbf{Y}) \doteq \mathbf{D}_{\mathbf{X}} \mathbf{Y} - \mathbf{D}_{\mathbf{Y}} \mathbf{X} - [\mathbf{X}, \mathbf{Y}], \quad (1.6)$$

with a corresponding N-adapted decomposition into

$$\begin{aligned} \mathbf{T}(\mathbf{X}, \mathbf{Y}) = & \{h\mathbf{T}(h\mathbf{X}, h\mathbf{Y}), h\mathbf{T}(h\mathbf{X}, v\mathbf{Y}), h\mathbf{T}(v\mathbf{X}, h\mathbf{Y}), h\mathbf{T}(v\mathbf{X}, v\mathbf{Y}), \\ & v\mathbf{T}(h\mathbf{X}, h\mathbf{Y}), v\mathbf{T}(h\mathbf{X}, v\mathbf{Y}), v\mathbf{T}(v\mathbf{X}, h\mathbf{Y}), v\mathbf{T}(v\mathbf{X}, v\mathbf{Y})\}. \end{aligned} \quad (1.7)$$

The nontrivial N-adapted coefficients  $\mathbf{T} = \{\mathbf{T}_{\beta\gamma}^\alpha = -\mathbf{T}_{\gamma\beta}^\alpha = (T_{jk}^i, T_{ja}^i, T_{jk}^a, T_{ja}^b, T_{ca}^b)\}$  are given in Refs. [3, 4].<sup>14</sup>

The curvature of a d-connection  $\mathbf{D}$  is defined

$$\mathbf{R}(\mathbf{X}, \mathbf{Y}) \doteq \mathbf{D}_{\mathbf{X}} \mathbf{D}_{\mathbf{Y}} - \mathbf{D}_{\mathbf{Y}} \mathbf{D}_{\mathbf{X}} - \mathbf{D}_{[\mathbf{X}, \mathbf{Y}]}, \quad (1.8)$$

with N-adapted decomposition

$$\begin{aligned} \mathbf{R}(\mathbf{X}, \mathbf{Y})\mathbf{Z} = & \{\mathbf{R}(h\mathbf{X}, h\mathbf{Y})h\mathbf{Z}, \mathbf{R}(h\mathbf{X}, v\mathbf{Y})h\mathbf{Z}, \mathbf{R}(v\mathbf{X}, h\mathbf{Y})h\mathbf{Z}, \\ & \mathbf{R}(v\mathbf{X}, v\mathbf{Y})h\mathbf{Z}, \mathbf{R}(h\mathbf{X}, h\mathbf{Y})v\mathbf{Z}, \mathbf{R}(h\mathbf{X}, v\mathbf{Y})v\mathbf{Z}, \\ & \mathbf{R}(v\mathbf{X}, h\mathbf{Y})v\mathbf{Z}, \mathbf{R}(v\mathbf{X}, v\mathbf{Y})v\mathbf{Z}\}. \end{aligned} \quad (1.9)$$

The formulas for local N-adapted components and their symmetries, of the d-torsion and d-curvature, can be computed by introducing  $\mathbf{X} = \mathbf{e}_\alpha$ ,  $\mathbf{Y} = \mathbf{e}_\beta$  and  $\mathbf{Z} = \mathbf{e}_\gamma$  in (1.9). The formulas for nontrivial N-adapted coefficients

$$\mathbf{R} = \{\mathbf{R}_{\beta\gamma\delta}^\alpha = (R_{hjk}^i, R_{bjk}^a, R_{hja}^i, R_{bj a}^c, R_{hba}^i, R_{bea}^c)\}$$

are given in [3, 4]. Contracting the first and forth indices  $\mathbf{R}_{\beta\gamma} = \mathbf{R}_{\beta\gamma\alpha}^\alpha$ , one gets the N-adapted coefficients for the Ricci tensor

$$\mathbf{Ric} \doteq \{\mathbf{R}_{\beta\gamma} = (R_{ij}, R_{ia}, R_{ai}, R_{ab})\}. \quad (1.10)$$

A distinguished metric (in brief, d-metric) on a N-anholonomic manifold  $\mathbf{V}$  is a second rank symmetric tensor  $\mathbf{g}$  which in N-adapted form is written

$$\mathbf{g} = g_{ij}(x, y) e^i \otimes e^j + g_{ab}(x, y) e^a \otimes e^b. \quad (1.11)$$

In brief, we write  $\mathbf{g} = hg \oplus_N vg = [{}^h g, {}^v g]$ . With respect to coordinate co-frames, the metric  $\mathbf{g}$  can be written in the form

$$\mathbf{g} = \underline{g}_{\alpha\beta}(u) du^\alpha \otimes du^\beta \quad (1.12)$$

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<sup>14</sup>We omit repeating of cumbersome local formulas but emphasize the h- and v-decomposition of geometrical objects which is important for our further constructions.

where

$$\underline{g}_{\alpha\beta} = \begin{bmatrix} g_{ij} + N_i^a N_j^b h_{ab} & N_j^e g_{ae} \\ N_i^e g_{be} & g_{ab} \end{bmatrix}. \quad (1.13)$$

A d-connection  $\mathbf{D}$  is compatible to a metric  $\mathbf{g}$  if  $\mathbf{D}\mathbf{g} = 0$ .

There are two classes of preferred linear connections defined by the coefficients  $\{\underline{g}_{\alpha\beta}\}$  of a metric structure  $\mathbf{g}$  (equivalently, by the coefficients of corresponding d-metric  $(g_{ij}, h_{ab})$  and N-connection  $N_i^a$  : we shall emphasize the functional dependence on such coefficients in some formulas):

- The unique metric compatible and torsionless Levi Civita connection  $\nabla = \{ {}_1\Gamma_{\alpha\beta}^\gamma(g_{ij}, h_{ab}, N_i^a) \}$ , for which  ${}_1T_{\beta\gamma}^\alpha = 0$  and  $\nabla\mathbf{g} = 0$ . This is not a d-connection because it does not preserve under parallelism the N-connection splitting (1.1). The curvature and Ricci tensors of  $\nabla$ , denoted  ${}_1R_{\beta\gamma\delta}^\alpha$  and  ${}_1R_{\beta\gamma}$ , are computed respectively by formulas (1.8) and (1.10) when  $\mathbf{D} \rightarrow \nabla$ .
- The unique metric canonical d-connection  $\hat{\mathbf{D}} = \{ \hat{\Gamma}_{\alpha\beta}^\gamma(g_{ij}, h_{ab}, N_i^a) \}$  is defined by the conditions  $\hat{\mathbf{D}}\mathbf{g} = 0$  and  $h\hat{\mathbf{T}}(hX, hY) = 0$  and  $v\hat{\mathbf{T}}(vX, vY) = 0$ . The N-adapted coefficients  $\hat{\Gamma}_{\alpha\beta}^\gamma = (\hat{L}_{jk}^i, \hat{L}_{bk}^a, \hat{C}_{jc}^i, \hat{C}_{bc}^a)$  and the deformation tensor  ${}_1Z_{\alpha\beta}^\gamma$ , when  $\nabla = \hat{\mathbf{D}} + \hat{Z}$ ,

$${}_1\Gamma_{\alpha\beta}^\gamma(g_{ij}, g_{ab}, N_i^a) = \hat{\Gamma}_{\alpha\beta}^\gamma(g_{ij}, g_{ab}, N_i^a) + {}_1Z_{\alpha\beta}^\gamma(g_{ij}, g_{ab}, N_i^a)$$

for

$$\begin{aligned} \hat{L}_{jk}^i &= \frac{1}{2} g^{ir} (e_k g_{jr} + e_j g_{kr} - e_r g_{jk}), \\ \hat{L}_{bk}^a &= e_b(N_k^a) + \frac{1}{2} g^{ac} (e_k g_{bc} - g_{dc} e_b N_k^d - g_{db} e_c N_k^d), \\ \hat{C}_{jc}^i &= \frac{1}{2} g^{ik} e_c g_{jk}, \quad \hat{C}_{bc}^a = \frac{1}{2} g^{ad} (e_c g_{bd} + e_b g_{cd} - e_d g_{bc}). \end{aligned} \quad (1.14)$$

and  $\hat{Z} = \{ {}_1Z_{\alpha\beta}^\gamma \}$  given in [3, 4].

We shall underline symbols or indices of geometrical objects in order to emphasize that the components/formulas/equations are written with respect to a local coordinate basis, for instance,  $\underline{g}_{\alpha\beta} = g_{\underline{\alpha}\underline{\beta}}$ ,  $\underline{\hat{\Gamma}}_{\alpha\beta}^\gamma = \hat{\Gamma}_{\underline{\alpha}\underline{\beta}}^\gamma$ ,  ${}_1\underline{\Gamma}_{\alpha\beta}^\gamma = {}_1\Gamma_{\underline{\alpha}\underline{\beta}}^\gamma$ ,  $\underline{\hat{\mathbf{R}}}_{\beta\gamma} = \hat{\mathbf{R}}_{\underline{\beta}\underline{\gamma}}, \dots$

Having prescribed a nonholonomic  $n + m$  splitting with coefficients  $N_i^a$  on a (semi) Riemannian manifold  $\mathbf{V}$  provided with metric structure  $\underline{g}_{\alpha\beta}$

(1.12), we can work with N-adapted frames (1.3) and (1.4) and the equivalent d-metric structure  $(g_{ij}, g_{ab})$  (1.11). On  $\mathbf{V}$ , one can be introduced two (equivalent) canonical metric compatible (both defined by the same metric structure, equivalently, by the same d-metric and N-connection) linear connections: the Levi Civita connection  $\nabla$  and the canonical d-connection  $\hat{\mathbf{D}}$ . In order to perform geometric constructions in N-adapted form, we have to work with the connection  $\hat{\mathbf{D}}$  which contains nontrivial torsion coefficients  $\hat{T}_{ja}^i, \hat{T}_{jk}^a, \hat{T}_{ja}^b$  induced by the "off diagonal" metric / N-connection coefficients  $N_i^a$  and their derivatives.

We conclude that the geometry of a N-aholonomic manifold  $\mathbf{V}$  can be described by data  $\{g_{ij}, g_{ab}, N_i^a, \nabla\}$  or, equivalently, by data  $\{g_{ij}, g_{ab}, N_i^a, \hat{\mathbf{D}}\}$ . Of course, two different linear connections, even defined by the same metric structure, are characterized by different Ricci and Riemann curvature tensors and curvature scalars. In this works, we shall prefer N-adapted constructions with  $\hat{\mathbf{D}}$  but also apply  $\nabla$  if the proofs for  $\hat{\mathbf{D}}$  will be cumbersome. The idea is that if a geometric Ricci flow construction is well defined for one of the connections,  $\nabla$  or  $\hat{\mathbf{D}}$ , it can be equivalently redefined for the second one by considering the distortion tensor  ${}_i Z_{\alpha\beta}^\gamma$ .

### On nonholonomic evolution equations

The Ricci flow equations were introduced by R. Hamilton as evolution equations

$$\frac{\partial g_{\alpha\beta}(\chi)}{\partial \chi} = -2 {}_i R_{\alpha\beta}(\chi) \quad (1.15)$$

for a set of Riemannian metrics  $g_{\alpha\beta}(\chi)$  and corresponding Ricci tensors  ${}_i R_{\alpha\beta}(\chi)$  parametrized by a real  $\chi$ .

The normalized (holonomic) Ricci flows, with respect to the coordinate base  $\partial_{\underline{\alpha}} = \partial/\partial u^{\underline{\alpha}}$ , are described by the equations

$$\frac{\partial}{\partial \chi} g_{\underline{\alpha}\underline{\beta}} = -2 {}_i R_{\underline{\alpha}\underline{\beta}} + \frac{2r}{5} g_{\underline{\alpha}\underline{\beta}}, \quad (1.16)$$

where the normalizing factor  $r = \int {}_i R dV/dV$  is introduced in order to preserve the volume  $V$ . For N-anholonomic Ricci flows, the coefficients  $g_{\underline{\alpha}\underline{\beta}}$  are parametrized in the form (1.13).

With respect to the N-adapted frames (1.3) and (1.4), when

$$\mathbf{e}_\alpha(\chi) = \mathbf{e}_\alpha^{\underline{\alpha}}(\chi) \partial_{\underline{\alpha}} \text{ and } \mathbf{e}^\alpha(\chi) = \mathbf{e}^{\underline{\alpha}}_\alpha(\chi) du^{\underline{\alpha}},$$

the frame transforms are respectively parametrized in the form

$$\begin{aligned} \mathbf{e}_\alpha^{\underline{a}}(\chi) &= \begin{bmatrix} e_i^{\underline{a}} = \delta_i^{\underline{a}} & e_i^{\underline{a}} = N_i^b(\chi) \delta_b^{\underline{a}} \\ e_a^{\underline{a}} = 0 & e_a^{\underline{a}} = \delta_a^{\underline{a}} \end{bmatrix}, \\ \mathbf{e}_{\underline{a}}^\alpha(\chi) &= \begin{bmatrix} e_i^{\underline{a}} = \delta_i^{\underline{a}} & e_i^{\underline{a}} = -N_k^b(\chi) \delta_b^{\underline{a}} \\ e_a^{\underline{a}} = 0 & e_a^{\underline{a}} = \delta_a^{\underline{a}} \end{bmatrix}, \end{aligned} \quad (1.17)$$

where  $\delta_{\underline{i}}^{\underline{i}}$  is the Kronecher symbol.

**Definition 1.2.1** *Nonholonomic deformations of geometric objects (and related systems of equations) on a N-anholonomic manifold  $\mathbf{V}$  are defined for the same metric structure  $\mathbf{g}$  by a set of transforms of arbitrary frames into N-adapted ones and of the Levi Civita connection  $\nabla$  into the canonical d-connection  $\hat{\mathbf{D}}$ , locally parametrized in the form*

$$\partial_{\underline{\alpha}} = (\partial_{\underline{i}}, \partial_{\underline{a}}) \rightarrow \mathbf{e}_\alpha = (\mathbf{e}_i, e_a); \quad g_{\underline{\alpha}\underline{\beta}} \rightarrow [g_{ij}, g_{ab}, N_i^a]; \quad {}_1\Gamma_{\alpha\beta}^\gamma \rightarrow \hat{\Gamma}_{\alpha\beta}^\gamma.$$

It should be noted that the heuristic arguments presented in this section do not provide a rigorous proof of evolution equations with  $\hat{\mathbf{D}}$  and  $\hat{\mathbf{R}}_{\alpha\beta}$  all defined with respect to N-adapted frames (1.3) and (1.4).<sup>15</sup> A rigorous proof for nonholonomic evolution equations is possible following a N-adapted variational calculus for the Perelman's functionals presented (below) for Theorems 1.2.1 and 1.2.2.

### Generalized Perelman's functionals

Following G. Perelman's ideas, the Ricci flow equations can be derived as gradient flows for some functionals defined by the Levi Civita connection  $\nabla$ . The functionals are written in the form (we use our system of denotations)

$$\begin{aligned} {}_1\mathcal{F}(\mathbf{g}, \nabla, f) &= \int_{\mathbf{V}} \left( {}_1R + |\nabla f|^2 \right) e^{-f} dV, \\ {}_1\mathcal{W}(\mathbf{g}, \nabla, f, \tau) &= \int_{\mathbf{V}} \left[ \tau ({}_1R + |\nabla f|^2) + f - (n + m) \right] \mu dV, \end{aligned} \quad (1.18)$$

where  $dV$  is the volume form of  $\mathbf{g}$ , integration is taken over compact  $\mathbf{V}$  and  ${}_1R$  is the scalar curvature computed for  $\nabla$ . For a parameter  $\tau > 0$ , we

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<sup>15</sup>The tensor  $\hat{\mathbf{R}}_{\alpha\beta}$  is not symmetric which results, in general, in Ricci flows of nonsymmetric metrics.

have  $\int_{\mathbf{V}} \mu dV = 1$  when  $\mu = (4\pi\tau)^{-(n+m)/2} e^{-f}$ . Following this approach, the Ricci flow is considered as a dynamical system on the space of Riemannian metrics and the functionals  ${}_1\mathcal{F}$  and  ${}_1\mathcal{W}$  are of Lyapunov type. Ricci flat configurations are defined as "fixed" on  $\tau$  points of the corresponding dynamical systems.

The functionals (1.18) can be also re-defined in equivalent form for the canonical d-connection, in the case of Lagrange-Finsler spaces. In this section, we show that the constructions can be generalized for arbitrary N-anholonomic manifolds, when the gradient flow is constrained to be adapted to the corresponding N-connection structure.

**Claim 1.2.1** *For a set of N-anholonomic manifolds of dimension  $n + m$ , the Perelman's functionals for the canonical d-connection  $\hat{\mathbf{D}}$  are defined*

$$\begin{aligned}\hat{\mathcal{F}}(\mathbf{g}, \hat{\mathbf{D}}, \hat{f}) &= \int_{\mathbf{V}} \left( {}^h\hat{R} + {}^v\hat{R} + |\hat{\mathbf{D}}\hat{f}|^2 \right) e^{-\hat{f}} dV, \\ \hat{\mathcal{W}}(\mathbf{g}, \hat{\mathbf{D}}, \hat{f}, \hat{\tau}) &= \int_{\mathbf{V}} [\hat{\tau}({}^h\hat{R} + {}^v\hat{R} + |{}^hD\hat{f}| + |{}^vD\hat{f}|)^2 + \hat{f} - (n+m)] \hat{\mu} dV,\end{aligned}\tag{1.19}$$

where  $dV$  is the volume form of  ${}^L\mathbf{g}$ ;  $R$  and  $S$  are respectively the  $h$ - and  $v$ -components of the curvature scalar of  $\hat{\mathbf{D}}$  when  ${}^s\hat{\mathbf{R}} \doteq \mathbf{g}^{\alpha\beta} \hat{\mathbf{R}}_{\alpha\beta} = {}^h\hat{R} + {}^v\hat{R}$ , for  $\hat{\mathbf{D}}_{\alpha} = (D_i, D_a)$ , or  $\hat{\mathbf{D}} = ({}^hD, {}^vD)$  when  $|\hat{\mathbf{D}}\hat{f}|^2 = |{}^hD\hat{f}|^2 + |{}^vD\hat{f}|^2$ , and  $\hat{f}$  satisfies  $\int_{\mathbf{V}} \hat{\mu} dV = 1$  for  $\hat{\mu} = (4\pi\tau)^{-(n+m)/2} e^{-\hat{f}}$  and  $\hat{\tau} > 0$ .

Elaborating an N-adapted variational calculus, we shall consider both variations in the so-called  $h$ - and  $v$ -subspaces stated by decompositions (1.1). For simplicity, we consider the  $h$ -variation  ${}^h\delta g_{ij} = v_{ij}$ , the  $v$ -variation  ${}^v\delta g_{ab} = v_{ab}$ , for a fixed N-connection structure in (1.11), and  ${}^h\delta \hat{f} = {}^h\delta f$ ,  ${}^v\delta \hat{f} = {}^v\delta f$ .

A number of important results in Riemannian geometry can be proved by using normal coordinates in a point  $u_0$  and its vicinity. Such constructions can be performed on a N-anholonomic manifold  $\mathbf{V}$ .

**Proposition 1.2.1** *For any point  $u_0 \in \mathbf{V}$ , there is a system of N-adapted coordinates for which  $\hat{\mathbf{\Gamma}}_{\alpha\beta}^{\gamma}(u_0) = 0$ .*

**Proof.** In the system of normal coordinates in  $u_0$ , for the Levi Civita connection, when  ${}_1\Gamma_{\alpha\beta}^{\gamma}(u_0) = 0$ , we chose  $\mathbf{e}_{\alpha}\mathbf{g}_{\beta\gamma}|_{u_0} = 0$ . Following formulas (1.14), for a d-metric (1.11), equivalently (1.12), we get  $\hat{\mathbf{\Gamma}}_{\alpha\beta}^{\gamma}(u_0) = 0$ .  $\square$

We generalize for arbitrary N-anholonomic manifolds (see proof in [4]):

**Lemma 1.2.1** *The first  $N$ -adapted variations of (1.19) are given by*

$$\begin{aligned} \delta \widehat{\mathcal{F}}(v_{ij}, v_{ab}, {}^h f, {}^v f) = & \quad (1.20) \\ & \int_{\mathbf{V}} \{ [-v^{ij}(\widehat{R}_{ij} + \widehat{D}_i \widehat{D}_j \widehat{f}) + (\frac{{}^h v}{2} - {}^h f) (2 {}^h \Delta \widehat{f} - |{}^h D \widehat{f}|^2) + {}^h \widehat{R}] \\ & + [-v^{ab}(\widehat{R}_{ab} + \widehat{D}_a \widehat{D}_b \widehat{f}) + (\frac{{}^v v}{2} - {}^v f) (2 {}^v \Delta \widehat{f} - |{}^v D \widehat{f}|^2) + {}^v \widehat{R}] \} e^{-\widehat{f}} dV \end{aligned}$$

where  ${}^h \Delta = \widehat{D}_i \widehat{D}^i$  and  ${}^v \Delta = \widehat{D}_a \widehat{D}^a$ , for  $\widehat{\Delta} = {}^h \Delta + {}^v \Delta$ , and  ${}^h v = g^{ij} v_{ij}$ ,  ${}^v v = g^{ab} v_{ab}$ .

**Definition 1.2.2** *A  $d$ -metric  $\mathbf{g}$  (1.11) evolving by the (nonholonomic) Ricci flow is called a (nonholonomic) breather if for some  $\chi_1 < \chi_2$  and  $\alpha > 0$  the metrics  $\alpha \mathbf{g}(\chi_1)$  and  $\alpha \mathbf{g}(\chi_2)$  differ only by a diffeomorphism (in the  $N$ -anholonomic case, preserving the Whitney sum (1.1)). The cases  $\alpha (=, <, >) > 1$  define correspondingly the (steady, shrinking) expanding breathers.*

The breather properties depend on the type of connections which are used for definition of Ricci flows. For  $N$ -anholonomic manifolds, one can be the situation when, for instance, the  $h$ -component of metric is steady but the  $v$ -component is shrinking.

### Main theorems on nonholonomic Ricci flows

Following a  $N$ -adapted variational calculus for  $\widehat{\mathcal{F}}(\mathbf{g}, \widehat{f})$ , see Lemma 1.2.1, with Laplacian  $\widehat{\Delta}$  and  $h$ - and  $v$ -components of the Ricci tensor,  $\widehat{R}_{ij}$  and  $\widehat{R}_{ab}$ , defined by  $\widehat{\mathbf{D}}$  and considering parameter  $\tau(\chi)$ ,  $\partial \tau / \partial \chi = -1$  (for simplicity, we shall not consider the normalized term and put  $\lambda = 0$ ), one holds

**Theorem 1.2.1** *The Ricci flows of  $d$ -metrics are characterized by evolution equations*

$$\begin{aligned} \frac{\partial g_{ij}}{\partial \chi} &= -2\widehat{R}_{ij}, \quad \frac{\partial g_{ab}}{\partial \chi} = -2\widehat{R}_{ab}, \\ \frac{\partial \widehat{f}}{\partial \chi} &= -\widehat{\Delta} \widehat{f} + |\widehat{\mathbf{D}} \widehat{f}|^2 - {}^h \widehat{R} - {}^v \widehat{R} \end{aligned}$$

and the property that

$$\frac{\partial}{\partial \chi} \widehat{\mathcal{F}}(\mathbf{g}(\chi), \widehat{f}(\chi)) = 2 \int_{\mathbf{V}} \left[ |\widehat{R}_{ij} + \widehat{D}_i \widehat{D}_j \widehat{f}|^2 + |\widehat{R}_{ab} + \widehat{D}_a \widehat{D}_b \widehat{f}|^2 \right] e^{-\widehat{f}} dV$$

and  $\int_{\mathbf{V}} e^{-\hat{f}} dV$  is constant. The functional  $\hat{\mathcal{F}}(\mathbf{g}(\chi), \hat{f}(\chi))$  is nondecreasing in time and the monotonicity is strict unless we are on a steady  $d$ -gradient solution.

On  $N$ -anholonomic manifolds, we define the associated  $d$ -energy

$$\hat{\lambda}(\mathbf{g}, \hat{\mathbf{D}}) \doteq \inf\{\hat{\mathcal{F}}(\mathbf{g}(\chi), \hat{f}(\chi)) \mid \hat{f} \in C^\infty(\mathbf{V}), \int_{\mathbf{V}} e^{-\hat{f}} dV = 1\}. \quad (1.21)$$

This value contains information on nonholonomic structure on  $\mathbf{V}$ . It is also possible to introduce the associated energy defined by  ${}_h\mathcal{F}(\mathbf{g}, \nabla, f)$  from (1.18),  $\lambda(\mathbf{g}, \nabla) \doteq \inf\{{}_h\mathcal{F}(\mathbf{g}(\chi), f(\chi)) \mid f \in C^\infty(\mathbf{V}), \int_{\mathbf{V}} e^{-f} dV = 1\}$ . Both

values  $\hat{\lambda}$  and  $\lambda$  are defined by the same sets of metric structures  $\mathbf{g}(\chi)$  but, respectively, for different sets of linear connections,  $\hat{\mathbf{D}}(\chi)$  and  $\nabla(\chi)$ . One holds also the property that  $\lambda$  is invariant under diffeomorphisms but  $\hat{\lambda}$  possesses only  $N$ -adapted diffeomorphism invariance. In this section, we state the main properties of  $\hat{\lambda}$ .

**Proposition 1.2.2** *There are canonical  $N$ -adapted decompositions, to splitting (1.1), of the functional  $\hat{\mathcal{F}}$  and associated  $d$ -energy  $\hat{\lambda}$ .*

From this Proposition, one follows

**Corollary 1.2.1** *The  $d$ -energy (respectively,  $h$ -energy or  $v$ -energy) has the property:*

- $\hat{\lambda}$  (respectively,  ${}^h\hat{\lambda}$  or  ${}^v\hat{\lambda}$ ) is nondecreasing along the  $N$ -anholonomic Ricci flow and the monotonicity is strict unless we are on a steady distinguished (respectively, horizontal or vertical) gradient soliton;
- a steady distinguished (horizontal or vertical) breather is necessarily a steady distinguished (respectively, horizontal or vertical) gradient solution.

For any positive numbers  ${}^ha$  and  ${}^va$ ,  $\hat{a} = {}^ha + {}^va$ , and  $N$ -adapted diffeomorphisms on  $\mathbf{V}$ , denoted  $\hat{\varphi} = ({}^h\varphi, {}^v\varphi)$ , we have

$$\widehat{\mathcal{W}}({}^ha {}^h\varphi^* g_{ij}, {}^va {}^v\varphi^* g_{ab}, \hat{\varphi}^* \hat{\mathbf{D}}, \hat{\varphi}^* \hat{f}, \hat{a}\hat{\tau}) = \widehat{\mathcal{W}}(\mathbf{g}, \hat{\mathbf{D}}, \hat{f}, \hat{\tau})$$

which mean that the functional  $\widehat{\mathcal{W}}$  is invariant under  $N$ -adapted parabolic scaling, i.e. under respective scaling of  $\hat{\tau}$  and  $\mathbf{g}_{\alpha\beta} = (g_{ij}, g_{ab})$ . For simplicity,

we can restrict our considerations to evolutions defined by d-metric coefficients  $\mathbf{g}_{\alpha\beta}(\hat{\tau})$  with not depending on  $\hat{\tau}$  values  $N_i^a(u^\beta)$ . In a similar form to Lemma 1.2.1, we get the following first N-adapted variation formula for  $\widehat{\mathcal{W}}$  :

**Lemma 1.2.2** *The first N-adapted variations of  $\widehat{\mathcal{W}}$  are given by*

$$\begin{aligned} \delta\widehat{\mathcal{W}}(v_{ij}, v_{ab}, {}^h f, {}^v f, \hat{\tau}) = \\ \int_{\mathbf{V}} \left\{ \hat{\tau} \left[ -v^{ij} (\hat{R}_{ij} + \hat{D}_i \hat{D}_j \hat{f} - \frac{g_{ij}}{2\hat{\tau}}) - v^{ab} (\hat{R}_{ab} + \hat{D}_a \hat{D}_b \hat{f} - \frac{g_{ab}}{2\hat{\tau}}) \right] \right. \\ + \left( \frac{{}^h v}{2} - {}^h f - \frac{n}{2\hat{\tau}} \hat{\eta} \right) [\hat{\tau} ({}^h \hat{R} + 2 {}^h \Delta \hat{f} - |{}^h D \hat{f}|^2) + {}^h f - n - 1] \\ + \left( \frac{{}^v v}{2} - {}^v f - \frac{m}{2\hat{\tau}} \hat{\eta} \right) [\hat{\tau} ({}^v \hat{R} + 2 {}^v \Delta \hat{f} - |{}^v D \hat{f}|^2) + {}^v f - m - 1] \\ \left. + \hat{\eta} \left( {}^h \hat{R} + {}^v \hat{R} + |{}^h D \hat{f}|^2 + |{}^v D \hat{f}|^2 - \frac{n+m}{2\hat{\tau}} \right) \right\} (4\pi\hat{\tau})^{-(n+m)/2} e^{-\hat{f}} dV, \end{aligned}$$

where  $\hat{\eta} = \delta\hat{\tau}$ .

For the functional  $\widehat{\mathcal{W}}$ , one holds a result which is analogous to Theorem 1.2.1:

**Theorem 1.2.2** *If a d-metric  $\mathbf{g}(\chi)$  (1.11) and functions  $\hat{f}(\chi)$  and  $\hat{\tau}(\chi)$  evolve according the system of equations*

$$\begin{aligned} \frac{\partial g_{ij}}{\partial \chi} &= -2\hat{R}_{ij}, \quad \frac{\partial g_{ab}}{\partial \chi} = -2\hat{R}_{ab}, \\ \frac{\partial \hat{f}}{\partial \chi} &= -\hat{\Delta} \hat{f} + |\hat{\mathbf{D}} \hat{f}|^2 - {}^h \hat{R} - {}^v \hat{R} + \frac{n+m}{\hat{\tau}}, \quad \frac{\partial \hat{\tau}}{\partial \chi} = -1 \end{aligned}$$

and the property that

$$\begin{aligned} \frac{\partial}{\partial \chi} \widehat{\mathcal{W}}(\mathbf{g}(\chi), \hat{f}(\chi), \hat{\tau}(\chi)) &= 2 \int_{\mathbf{V}} \hat{\tau} [|\hat{R}_{ij} + D_i D_j \hat{f} - \frac{1}{2\hat{\tau}} g_{ij}|^2 + \\ &\quad |\hat{R}_{ab} + D_a D_b \hat{f} - \frac{1}{2\hat{\tau}} g_{ab}|^2] (4\pi\hat{\tau})^{-(n+m)/2} e^{-\hat{f}} dV \end{aligned}$$

and  $\int_{\mathbf{V}} (4\pi\hat{\tau})^{-(n+m)/2} e^{-\hat{f}} dV$  is constant. The functional  $\widehat{\mathcal{W}}$  is  $h-$  ( $v-$ ) nondecreasing in time and the monotonicity is strict unless we are on a shrinking  $h-$  ( $v-$ ) gradient soliton. This functional is N-adapted nondecreasing if it is both  $h-$  and  $v-$  nondecreasing.



In this work, for Theorem 1.2.2, the evolution equations are written with respect to N-adapted frames. If the N-connection structure is fixed in "time"  $\chi$ , or  $\hat{\tau}$ , we do not have to consider evolution equations for the N-anholonomic frame structure. For more general cases, the evolution of preferred N-adapted frames (1.17) is stated by:

**Corollary 1.2.2** *The evolution, for all time  $\tau \in [0, \tau_0)$ , of preferred frames on a N-anholonomic manifold  $\mathbf{e}_\alpha(\tau) = \mathbf{e}_\alpha^{\underline{a}}(\tau, u) \partial_{\underline{a}}$  is defined by*

$$\begin{aligned} \mathbf{e}_\alpha^{\underline{a}}(\tau, u) &= \begin{bmatrix} e_i^{\underline{a}}(\tau, u) & N_i^{\underline{b}}(\tau, u) e_b^{\underline{a}}(\tau, u) \\ 0 & e_a^{\underline{a}}(\tau, u) \end{bmatrix}, \\ \mathbf{e}_{\underline{a}}^\alpha(\tau, u) &= \begin{bmatrix} e^i_{\underline{a}} = \delta_{\underline{a}}^i & e^b_{\underline{a}} = -N_k^b(\tau, u) \delta_{\underline{a}}^k \\ e^i_{\underline{a}} = 0 & e^a_{\underline{a}} = \delta_{\underline{a}}^a \end{bmatrix}, \end{aligned}$$

with  $g_{ij}(\tau) = e_i^{\underline{a}}(\tau, u) e_j^{\underline{b}}(\tau, u) \eta_{\underline{a}\underline{b}}$  and  $g_{ab}(\tau) = e_a^{\underline{a}}(\tau, u) e_b^{\underline{b}}(\tau, u) \eta_{\underline{a}\underline{b}}$ , where  $\eta_{\underline{a}\underline{b}} = \text{diag}[\pm 1, \dots, \pm 1]$  and  $\eta_{ab} = \text{diag}[\pm 1, \dots, \pm 1]$  establish the signature of  $\mathbf{g}_{\alpha\beta}^{[0]}(u)$ , is given by equations  $\frac{\partial}{\partial \tau} \mathbf{e}_{\underline{a}}^\alpha = \mathbf{g}^{\alpha\beta} \hat{\mathbf{R}}_{\beta\gamma} \mathbf{e}_{\underline{a}}^\gamma$  if we prescribe that the geometric constructions are derived by the canonical d-connection.

It should be noted that  $\mathbf{g}^{\alpha\beta} \hat{\mathbf{R}}_{\beta\gamma} = g^{ij} \hat{R}_{ij} + g^{ab} \hat{R}_{ab}$  selects for evolution only the symmetric components of the Ricci d-tensor for the canonical d-connection.

### 1.2.2 Clifford structures adapted to nonlinear connections

In this section, we outline some key results from Refs. [1, 6] on spinors and Dirac operators for nonholonomic manifolds and generalized Finsler spaces.

The spinor bundle on a manifold  $M$ ,  $\dim M = n$ , is constructed on the tangent bundle  $TM$  by substituting the group  $SO(n)$  by its universal covering  $Spin(n)$ . If a horizontal quadratic form  ${}^h g_{ij}(x, y)$  is defined on  $T_x h\mathbf{V}$  we can consider h-spinor spaces in every point  $x \in h\mathbf{V}$  with fixed  $y^a$ . The constructions can be completed on  $T\mathbf{V}$  by using the d-metric  $\mathbf{g}$ . In this case, the group  $SO(n+m)$  is not only substituted by  $Spin(n+m)$  but with respect to N-adapted frames there are emphasized decompositions to  $Spin(n) \oplus Spin(m)$ .<sup>16</sup>

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<sup>16</sup>It should be noted here that spin bundles may not exist for general holonomic or nonholonomic manifolds. For simplicity, we do not provide such topological considerations in this paper. We state that we shall work only with N-anholonomic manifolds for which certain spinor structures can be defined both for the h- and v-splitting; the existence of a well defined decomposition  $Spin(n) \oplus Spin(m)$  follows from N-connection splitting.

### Clifford N-adapted modules (d-modules)

A Clifford d-algebra is a  $\wedge V^{n+m}$  algebra endowed with a product  $\mathbf{u}\mathbf{v} + \mathbf{v}\mathbf{u} = 2\mathbf{g}(\mathbf{u}, \mathbf{v}) \mathbb{I}$  distinguished into h-, v-products

$${}^h u {}^h v + {}^h v {}^h u = 2 {}^h g(u, v) {}^h \mathbb{I}, \quad {}^v u {}^v v + {}^v v {}^v u = 2 {}^v h({}^v u, {}^v v) {}^v \mathbb{I},$$

for any  $\mathbf{u} = ({}^h u, {}^v u)$ ,  $\mathbf{v} = ({}^h v, {}^v v) \in V^{n+m}$ , where  $\mathbb{I}$ ,  ${}^h \mathbb{I}$  and  ${}^v \mathbb{I}$  are unity matrices of corresponding dimensions  $(n+m) \times (n+m)$ , or  $n \times n$  and  $m \times m$ .

A metric  ${}^h g$  on  $h\mathbf{V}$  is defined by sections of the tangent space  $T h\mathbf{V}$  provided with a bilinear symmetric form on continuous sections  $\Gamma(T h\mathbf{V})$ .<sup>17</sup> This allows us to define Clifford h-algebras  ${}^h Cl(T_x h\mathbf{V})$ , in any point  $x \in T h\mathbf{V}$ ,  $\gamma_i \gamma_j + \gamma_j \gamma_i = 2 g_{ij} {}^h \mathbb{I}$ . For any point  $x \in h\mathbf{V}$  and fixed  $y = y_0$ , there exists a standard complexification,  $T_x h\mathbf{V}^{\mathbb{C}} \doteq T_x h\mathbf{V} + i T_x h\mathbf{V}$ , which can be used for definition of the 'involution' operator on sections of  $T_x h\mathbf{V}^{\mathbb{C}}$ ,

$${}^h \sigma_1 {}^h \sigma_2(x) \doteq {}^h \sigma_2(x) {}^h \sigma_1(x), \quad {}^h \sigma^*(x) \doteq {}^h \sigma(x)^*, \quad \forall x \in h\mathbf{V},$$

where  ${}^{**}$  denotes the involution on every  ${}^h Cl(T_x h\mathbf{V})$ .

**Definition 1.2.3** A Clifford d-space on a nonholonomic manifold  $\mathbf{V}$  enabled with a d-metric  $\mathbf{g}(x, y)$  and a N-connection  $\mathbf{N}$  is defined as a Clifford bundle  $Cl(\mathbf{V}) = {}^h Cl(h\mathbf{V}) \oplus {}^v Cl(v\mathbf{V})$ , for the Clifford h-space  ${}^h Cl(h\mathbf{V}) \doteq {}^h Cl(T^* h\mathbf{V})$  and Clifford v-space  ${}^v Cl(v\mathbf{V}) \doteq {}^v Cl(T^* v\mathbf{V})$ .

For a fixed N-connection structure, a Clifford N-anholonomic bundle on  $\mathbf{V}$  is defined  ${}^N Cl(\mathbf{V}) \doteq {}^N Cl(T^* \mathbf{V})$ . Let us consider a complex vector bundle  ${}^E \pi : E \rightarrow \mathbf{V}$  on an N-anholonomic space  $\mathbf{V}$  when the N-connection structure is given for the base manifold. The Clifford d-module of a vector bundle  $E$  is defined by the  $C(\mathbf{V})$ -module  $\Gamma(E)$  of continuous sections in  $E$ ,  $c : \Gamma({}^N Cl(\mathbf{V})) \rightarrow \text{End}(\Gamma(E))$ .

In general, a vector bundle on a N-anholonomic manifold may be not adapted to the N-connection structure on base space.

### h-spinors, v-spinors and d-spinors

Let us consider a vector space  $V^n$  provided with Clifford structure. We denote such a space  ${}^h V^n$  in order to emphasize that its tangent space is provided with a quadratic form  ${}^h g$ . We also write  ${}^h Cl(V^n) \equiv Cl({}^h V^n)$  and use subgroup  $SO({}^h V^n) \subset O({}^h V^n)$ .

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<sup>17</sup>for simplicity, we shall consider only "horizontal" geometric constructions if they are similar to "vertical" ones

**Definition 1.2.4** *The space of complex  $h$ -spins is defined by the subgroup*

$${}^h Spin^c(n) \equiv Spin^c({}^h V^n) \equiv {}^h Spin^c(V^n) \subset Cl({}^h V^n),$$

*determined by the products of pairs of vectors  $w \in {}^h V^{\mathbb{C}}$  when  $w \doteq \lambda u$  where  $\lambda$  is a complex number of module 1 and  $u$  is of unity length in  ${}^h V^n$ .*

Similar constructions can be performed for the  $v$ -subspace  ${}^v V^m$ , which allows us to define similarly the group of real  $v$ -spins.

A usual spinor is a section of a vector bundle  $S$  on a manifold  $M$  when an irreducible representation of the group  $Spin(M) \doteq Spin(T_x^* M)$  is defined on the typical fiber. The set of sections  $\Gamma(S)$  is a irreducible Clifford module. If the base manifold is of type  $h\mathbf{V}$ , or is a general  $N$ -anholonomic manifold  $\mathbf{V}$ , we have to define spinors on such spaces in a form adapted to the respective  $N$ -connection structure.

**Definition 1.2.5** *A  $h$ -spinor bundle  ${}^h S$  on a  $h$ -space  $h\mathbf{V}$  is a complex vector bundle with both defined action of the  $h$ -spin group  ${}^h Spin(V^n)$  on the typical fiber and an irreducible representation of the group  ${}^h Spin(\mathbf{V}) \equiv Spin(h\mathbf{V}) \doteq Spin(T_x^* h\mathbf{V})$ . The set of sections  $\Gamma({}^h S)$  defines an irreducible Clifford  $h$ -module.*

The concept of "d-spinors" has been introduced for the spaces provided with  $N$ -connection structure [1]:

**Definition 1.2.6** *A distinguished spinor ( $d$ -spinor) bundle  $\mathbf{S} \doteq ({}^h S, {}^v S)$  on a  $N$ -anholonomic manifold  $\mathbf{V}$ ,  $\dim \mathbf{V} = n+m$ , is a complex vector bundle with a defined action of the spin  $d$ -group  $Spin \mathbf{V} \doteq Spin(V^n) \oplus Spin(V^m)$  with the splitting adapted to the  $N$ -connection structure which results in an irreducible representation  $Spin(\mathbf{V}) \doteq Spin(T^* \mathbf{V})$ . The set of sections  $\Gamma(\mathbf{S}) = \Gamma({}^h S) \oplus \Gamma({}^v S)$  is an irreducible Clifford  $d$ -module.*

If we study algebras through their representations, we also have to consider various algebras related by the Morita equivalence.<sup>18</sup>

The possibility to distinguish the  $Spin(n)$  (or, correspondingly  $Spin(h\mathbf{V})$ ,  $Spin(V^n) \oplus Spin(V^m)$ ) allows us to define an antilinear bijection  ${}^h J : {}^h S \rightarrow {}^h S$  (or  ${}^v J : {}^v S \rightarrow {}^v S$  and  $\mathbf{J} : \mathbf{S} \rightarrow \mathbf{S}$ ) with properties

$$\begin{aligned} {}^h J({}^h a \psi) &= {}^h \chi({}^h a) {}^h J {}^h \psi, \text{ for } {}^h a \in \Gamma^\infty(Cl(h\mathbf{V})); \\ ({}^h J {}^h \phi | {}^h J {}^h \psi) &= ({}^h \psi | {}^h \phi) \text{ for } {}^h \phi, {}^h \psi \in {}^h S. \end{aligned} \quad (1.22)$$

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<sup>18</sup>The Morita equivalence can be analyzed by applying in  $N$ -adapted form, both on the base and fiber spaces, the consequences of the Plymen's theorem (in this work, we omit details of such considerations).

The considerations presented in this Section consists the proof of:

**Theorem 1.2.3** *Any  $d$ -metric and  $N$ -connection structure defines naturally the fundamental geometric objects and structures (such as the Clifford  $h$ -module,  $v$ -module and Clifford  $d$ -modules, or the  $h$ -spin,  $v$ -spin structures and  $d$ -spinors) for the corresponding nonholonomic spin manifold and/or  $N$ -anholonomic spinor ( $d$ -spinor) manifold.*

### 1.2.3 $N$ -anholonomic Dirac operators

The geometric constructions depend on the type of linear connections considered for definition of such Dirac operators. They are metric compatible and  $N$ -adapted if the canonical  $d$ -connection is used (similar constructions can be performed for any deformation which results in a metric compatible  $d$ -connection).

#### Noholonomic vielbeins and spin $d$ -connections

Let us consider a Hilbert space of finite dimension. For a local dual coordinate basis  $e^{\underline{i}} \doteq dx^{\underline{i}}$  on  $h\mathbf{V}$ , we may respectively introduce certain classes of orthonormalized vielbeins and the  $N$ -adapted vielbeins,  $e^{\hat{i}} \doteq e^{\hat{i}}_{\underline{i}}(x, y) e^{\underline{i}}$  and  $e^{\hat{i}} \doteq e^{\hat{i}}_{\underline{i}}(x, y) e^{\underline{i}}$ , when  $g^{\underline{i}\underline{j}} e^{\hat{i}}_{\underline{i}} e^{\hat{j}}_{\underline{j}} = \delta^{\underline{i}\underline{j}}$  and  $g^{\underline{i}\underline{j}} e^{\hat{i}}_{\underline{i}} e^{\hat{j}}_{\underline{j}} = g^{\underline{i}\underline{j}}$ .

We define the algebra of Dirac's gamma horizontal matrices (in brief, gamma  $h$ -matrices defined by self-adjoint matrices  $M_k(\mathbb{C})$  where  $k = 2^{n/2}$  is the dimension of the irreducible representation of  $Cl(h\mathbf{V})$  from relation  $\gamma^{\hat{i}}\gamma^{\hat{j}} + \gamma^{\hat{j}}\gamma^{\hat{i}} = 2\delta^{\hat{i}\hat{j}} h\mathbb{I}$ . The action of  $dx^{\hat{i}} \in Cl(h\mathbf{V})$  on a spinor  ${}^h\psi \in {}^hS$  is given by formulas

$${}^h c(dx^{\hat{i}}) \doteq \gamma^{\hat{i}} \text{ and } {}^h c(dx^{\hat{i}}) {}^h\psi \doteq \gamma^{\hat{i}} {}^h\psi \equiv e^{\hat{i}}_{\underline{i}} \gamma^{\underline{i}} {}^h\psi. \quad (1.23)$$

Similarly, we can define the algebra of Dirac's gamma vertical matrices related to a typical fiber  $F$  (in brief, gamma  $v$ -matrices defined by self-adjoint matrices  $M'_k(\mathbb{C})$ , where  $k' = 2^{m/2}$  is the dimension of the irreducible representation of  $Cl(F)$ ) from relation  $\gamma^{\hat{a}}\gamma^{\hat{b}} + \gamma^{\hat{b}}\gamma^{\hat{a}} = 2\delta^{\hat{a}\hat{b}} v\mathbb{I}$ . The action of  $dy^{\hat{a}} \in Cl(F)$  on a spinor  ${}^v\psi \in {}^vS$  is  ${}^v c(dy^{\hat{a}}) := \gamma^{\hat{a}}$  and  ${}^v c(dy^{\hat{a}}) {}^v\psi \doteq \gamma^{\hat{a}} {}^v\psi \equiv e^{\hat{a}}_{\underline{a}} \gamma^{\underline{a}} {}^v\psi$ .

A more general gamma matrix calculus with distinguished gamma matrices (in brief, gamma  $d$ -matrices<sup>19</sup>) can be elaborated for  $N$ -anholonomic manifolds  $\mathbf{V}$  provided with  $d$ -metric structure  $\mathbf{g} = [{}^h g \oplus {}^v h]$  and for  $d$ -spinors  $\check{\psi} \doteq ({}^h\psi, {}^v\psi) \in \mathbf{S} \doteq ({}^hS, {}^vS)$ . In this case, we consider  $d$ -gamma

<sup>19</sup>in some our works we wrote  $\sigma$  instead of  $\gamma$

matrix relations  $\gamma^{\hat{\alpha}}\gamma^{\hat{\beta}} + \gamma^{\hat{\beta}}\gamma^{\hat{\alpha}} = 2\delta^{\hat{\alpha}\hat{\beta}} \mathbb{I}$ , with the action of  $du^\alpha \in \mathcal{Cl}(\mathbf{V})$  on a d-spinor  $\check{\psi} \in \mathbf{S}$  resulting in distinguished irreducible representations

$$\mathbf{c}(du^{\hat{\alpha}}) \doteq \gamma^{\hat{\alpha}} \text{ and } \mathbf{c} = (du^\alpha) \check{\psi} \doteq \gamma^\alpha \check{\psi} \equiv e^\alpha_{\hat{\alpha}} \gamma^{\hat{\alpha}} \check{\psi} \quad (1.24)$$

which allows us to write  $\gamma^\alpha(u)\gamma^\beta(u) + \gamma^\beta(u)\gamma^\alpha(u) = 2g^{\alpha\beta}(u) \mathbb{I}$ .

In the canonical representation, we have the irreducible form  $\check{\gamma} \doteq {}^h\gamma \oplus {}^v\gamma$  and  $\check{\psi} \doteq {}^h\psi \oplus {}^v\psi$ , for instance, by using block type of h- and v-matrices. We can also write such formulas as couples of gamma and/or h- and v-spinor objects written in N-adapted form,  $\gamma^\alpha \doteq ({}^h\gamma^i, {}^v\gamma^a)$  and  $\check{\psi} \doteq ({}^h\psi, {}^v\psi)$ .

The spin connection  ${}_S\nabla$  for Riemannian manifolds is induced by the Levi-Civita connection  $\Gamma$ ,  ${}_S\nabla \doteq d - \frac{1}{4} \Gamma^i_{jk} \gamma_i \gamma^j dx^k$ . On N-anholonomic manifolds, spin d-connection operators  ${}_S\hat{\nabla}$  can be similarly constructed from any metric compatible d-connection  $\mathbf{\Gamma}^\alpha_{\beta\mu}$  using the N-adapted absolute differential  $\delta$  acting, for instance, on a scalar function  $f(x, y)$  in the form  $\delta f = (\mathbf{e}_\nu f) \delta u^\nu = (\mathbf{e}_i f) dx^i + (e_a f) \delta y^a$ , for  $\delta u^\nu = \mathbf{e}^\nu$ , see N-elongated operators.

**Definition 1.2.7** *The canonical spin d-connection is defined by the canonical d-connection,*

$${}_S\hat{\nabla} \doteq \delta - \frac{1}{4} \hat{\mathbf{\Gamma}}^\alpha_{\beta\mu} \gamma_\alpha \gamma^\beta \delta u^\mu, \quad (1.25)$$

where the N-adapted coefficients  $\hat{\mathbf{\Gamma}}^\alpha_{\beta\mu}$  are given by formulas (1.14).

We note that the canonical spin d-connection  ${}_S\hat{\nabla}$  is metric compatible and contains nontrivial d-torsion coefficients induced by the N-anholonomy relations.

### Dirac d-operators

We consider a vector bundle  $\mathbf{E}$  on a N-anholonomic manifold  $\mathbf{V}$  (with two compatible N-connections defined as h- and v-splitting of  $T\mathbf{E}$  and  $T\mathbf{V}$ ). A d-connection  $\mathcal{D} : \Gamma^\infty(\mathbf{E}) \rightarrow \Gamma^\infty(\mathbf{E}) \otimes \Omega^1(\mathbf{V})$  preserves by parallelism splitting of the tangent total and base spaces and satisfy the Leibniz condition  $\mathcal{D}(f\sigma) = f(\mathcal{D}\sigma) + \delta f \otimes \sigma$ , for any  $f \in C^\infty(\mathbf{V})$ , and  $\sigma \in \Gamma^\infty(\mathbf{E})$  and  $\delta$  defining an N-adapted exterior calculus by using N-elongated operators which emphasize d-forms instead of usual forms on  $\mathbf{V}$ , with the coefficients taking values in  $\mathbf{E}$ .

The metricity and Leibniz conditions for  $\mathcal{D}$  are written respectively

$$\mathbf{g}(\mathcal{D}\mathbf{X}, \mathbf{Y}) + \mathbf{g}(\mathbf{X}, \mathcal{D}\mathbf{Y}) = \delta[\mathbf{g}(\mathbf{X}, \mathbf{Y})], \quad (1.26)$$

for any  $\mathbf{X}, \mathbf{Y} \in \chi(\mathbf{V})$ , and  $\mathcal{D}(\sigma\beta) \doteq \mathcal{D}(\sigma)\beta + \sigma\mathcal{D}(\beta)$ , for any  $\sigma, \beta \in \Gamma^\infty(\mathbf{E})$ . For local computations, we may define the corresponding coefficients of the geometric d-objects and write

$$\mathcal{D}\sigma_\beta \doteq \Gamma_{\beta\mu}^\alpha \sigma_\alpha \otimes \delta u^\mu = \Gamma_{\beta i}^\alpha \sigma_\alpha \otimes dx^i + \Gamma_{\beta a}^\alpha \sigma_\alpha \otimes \delta y^a,$$

where fiber "acute" indices are considered as spinor ones.

The respective actions of the Clifford d-algebra and Clifford h-algebra can be transformed into maps  $\Gamma^\infty(\mathbf{S}) \otimes \Gamma(Cl(\mathbf{V}))$  and  $\Gamma^\infty({}^h\mathbf{S}) \otimes \Gamma(Cl({}^h\mathbf{V}))$  to  $\Gamma^\infty(\mathbf{S})$  and, respectively,  $\Gamma^\infty({}^h\mathbf{S})$  by considering maps of type (1.23) and (1.24),  $\widehat{\mathbf{c}}(\check{\psi} \otimes \mathbf{a}) \doteq \mathbf{c}(\mathbf{a})\check{\psi}$  and  ${}^h\widehat{\mathbf{c}}({}^h\psi \otimes {}^ha) \doteq {}^hc({}^ha){}^h\psi$ .

**Definition 1.2.8** *The Dirac d-operator (Dirac h-operator, or v-operant) on a spin N-anholonomic manifold  $(\mathbf{V}, \mathbf{S}, J)$  (on a h-spin manifold  $(h\mathbf{V}, {}^h\mathbf{S}, {}^hJ)$ , or on a v-spin manifold  $(v\mathbf{V}, {}^v\mathbf{S}, {}^vJ)$ ) is defined*

$$\mathbb{D} := -i (\widehat{\mathbf{c}} \circ \mathbf{s}\nabla) = ({}^h\mathbb{D} = -i ({}^h\widehat{\mathbf{c}} \circ {}^h\mathbf{s}\nabla), {}^v\mathbb{D} = -i ({}^v\widehat{\mathbf{c}} \circ {}^v\mathbf{s}\nabla)) \quad (1.27)$$

Such N-adapted Dirac d-operators are called canonical and denoted  $\widehat{\mathbb{D}} = ({}^h\widehat{\mathbb{D}}, {}^v\widehat{\mathbb{D}})$  if they are defined for the canonical d-connection (1.14) and respective spin d-connection (1.25).

We formulate:

**Theorem 1.2.4** *Let  $(\mathbf{V}, \mathbf{S}, \mathbf{J})$  ( $(h\mathbf{V}, {}^h\mathbf{S}, {}^hJ)$ ) be a spin N-anholonomic manifold (h-spin space). There is the canonical Dirac d-operator (Dirac h-operator) defined by the almost Hermitian spin d-operator  $\mathbf{s}\widehat{\nabla} : \Gamma^\infty(\mathbf{S}) \rightarrow \Gamma^\infty(\mathbf{S}) \otimes \Omega^1(\mathbf{V})$  (spin h-operator  ${}^h\mathbf{s}\widehat{\nabla} : \Gamma^\infty({}^h\mathbf{S}) \rightarrow \Gamma^\infty({}^h\mathbf{S}) \otimes \Omega^1(h\mathbf{V})$ ) commuting with  $\mathbf{J}$  ( ${}^hJ$ ), see (1.22), and satisfying the conditions*

$$(\mathbf{s}\widehat{\nabla}\check{\psi} | \check{\phi}) + (\check{\psi} | \mathbf{s}\widehat{\nabla}\check{\phi}) = \delta(\check{\psi} | \check{\phi}) \text{ and } \mathbf{s}\widehat{\nabla}(\mathbf{c}(\mathbf{a})\check{\psi}) = \mathbf{c}(\widehat{\mathbf{D}}\mathbf{a})\check{\psi} + \mathbf{c}(\mathbf{a})\mathbf{s}\widehat{\nabla}\check{\psi}$$

for  $\mathbf{a} \in Cl(\mathbf{V})$  and  $\check{\psi} \in \Gamma^\infty(\mathbf{S})$ ,  $(({}^h\mathbf{s}\widehat{\nabla} {}^h\psi | {}^h\phi) + ({}^h\psi | {}^h\mathbf{s}\widehat{\nabla} {}^h\phi) = {}^h\delta({}^h\psi | {}^h\phi)$  and  ${}^h\mathbf{s}\widehat{\nabla}({}^hc({}^ha){}^h\psi) = {}^hc({}^h\widehat{D}{}^ha){}^h\psi + {}^hc({}^ha){}^h\mathbf{s}\widehat{\nabla} {}^h\psi$  for  ${}^ha \in Cl(h\mathbf{V})$  and  $\check{\psi} \in \Gamma^\infty({}^h\mathbf{S})$ ) determined by the metricity (1.26) and Leibnitz (??) conditions.

The geometric information of a spin manifold (in particular, the metric) is contained in the Dirac operator. For nonholonomic manifolds, the canonical Dirac d-operator has h- and v-irreducible parts related to off-diagonal metric terms and nonholonomic frames with associated structure. In a more special case, the canonical Dirac d-operator is defined by the canonical d-connection. Nonholonomic Dirac d-operators contain more information than the usual, holonomic, ones.

**Proposition 1.2.3** *If  $\widehat{\mathbb{D}} = ({}^h\widehat{\mathbb{D}}, {}^v\widehat{\mathbb{D}})$  is the canonical Dirac d-operator then  $[\widehat{\mathbb{D}}, f] = ic(\delta f)$ , equivalently,  $[{}^h\widehat{\mathbb{D}}, f] + [{}^v\widehat{\mathbb{D}}, f] = i({}^hc(dx^i \frac{\delta f}{\delta x^i}) + i({}^vc(\delta y^a \frac{\partial f}{\partial y^a}))$ , for all  $f \in C^\infty(\mathbf{V})$ .*

**Proof.** It is a straightforward computation following from Definition 1.2.8.  $\square$

The canonical Dirac d-operator and its h- and v-components have all the properties of the usual Dirac operators (for instance, they are self-adjoint but unbounded). It is possible to define a scalar product on  $\Gamma^\infty(\mathbf{S})$ ,

$$\langle \check{\psi}, \check{\phi} \rangle \doteq \int_{\mathbf{V}} (\check{\psi} | \check{\phi}) |\nu_{\mathbf{g}}| \quad (1.28)$$

where  $\nu_{\mathbf{g}} = \sqrt{\det|g|} \sqrt{\det|h|} dx^1 \dots dx^n dy^{n+1} \dots dy^{n+m}$  is the volume d-form on the N-anholonomic manifold  $\mathbf{V}$ .

### N-adapted spectral triples and distance in d-spinor spaces

We denote  ${}^N\mathcal{H} \doteq L_2(\mathbf{V}, \mathbf{S}) = [{}^h\mathcal{H} = L_2({}^h\mathbf{V}, {}^hS), {}^v\mathcal{H} = L_2({}^v\mathbf{V}, {}^vS)]$  the Hilbert d-space obtained by completing  $\Gamma^\infty(\mathbf{S})$  with the norm defined by the scalar product (1.28). Similarly to the holonomic spaces, by using formulas (1.27) and (1.25), one may prove that there is a self-adjoint unitary endomorphism  ${}_{[cr]}\Gamma$  of  ${}^N\mathcal{H}$ , called "chirality", being a  $\mathbb{Z}_2$  graduation of  ${}^N\mathcal{H}$ ,<sup>20</sup> which satisfies the condition  $\widehat{\mathbb{D}} {}_{[cr]}\Gamma = - {}_{[cr]}\Gamma \widehat{\mathbb{D}}$ . Such conditions can be written also for the irreducible components  ${}^h\widehat{\mathbb{D}}$  and  ${}^v\widehat{\mathbb{D}}$ .

**Definition 1.2.9** *A distinguished canonical spectral triple (canonical spectral d-triple)  $({}^N\mathcal{A}, {}^N\mathcal{H}, \widehat{\mathbb{D}})$  for a d-algebra  ${}^N\mathcal{A}$  is defined by a Hilbert d-space  ${}^N\mathcal{H}$ , a representation of  ${}^N\mathcal{A}$  in the algebra  ${}^N\mathcal{B}({}^N\mathcal{H})$  of d-operators bounded on  ${}^N\mathcal{H}$ , and by a self-adjoint d-operator  $\widehat{\mathbb{D}}$  on  ${}^N\mathcal{H}$ , of compact resolution,<sup>21</sup> such that  $[{}^N\mathcal{H}, a] \in {}^N\mathcal{B}({}^N\mathcal{H})$  for any  $a \in {}^N\mathcal{A}$ .*

Every canonical spectral d-triple is defined by two usual spectral triples which in our case corresponds to certain h- and v-components induced by the corresponding h- and v-components of the Dirac d-operator. For such spectral h(v)-triples we, can define the notion of  $KR^n$ -cycle and  $KR^m$ -cycle

<sup>20</sup>we use the label  $[cr]$  in order to avoid misunderstanding with the symbol  $\Gamma$  used for linear connections

<sup>21</sup>An operator  $D$  is of compact resolution if for any  $\lambda \in sp(D)$  the operator  $(D - \lambda \mathbb{I})^{-1}$  is compact.

and consider respective Hochschild complexes. To define a noncommutative geometry the  $h$ - and  $v$ - components of a canonical spectral  $d$ -triples must satisfy certain well defined seven conditions: the spectral dimensions are of order  $1/n$  and  $1/m$ , respectively, for  $h$ - and  $v$ -components of the canonical Dirac  $d$ -operator; there are satisfied the criteria of regularity, finiteness and reality; representations are of 1st order; there is orientability and Poincaré duality holds true. Such conditions can be satisfied by any Dirac operators and canonical Dirac  $d$ -operators (in the last case we have to work with  $d$ -objects). <sup>22</sup>

**Definition 1.2.10** *A spectral  $d$ -triple is a real one satisfying the above mentioned seven conditions for the  $h$ - and  $v$ -irreversible components and defining a ( $d$ -spinor)  $N$ -anholonomic noncommutative geometry stated by the data  $({}^N\mathcal{A}, {}^N\mathcal{H}, \widehat{\mathbb{D}}, \mathbf{J}, {}_{[cr]}\Gamma)$  and derived for the Dirac  $d$ -operator (1.27).*

For  $N$ -adapted constructions, we can consider  $d$ -algebras  ${}^N\mathcal{A} = {}^h\mathcal{A} \oplus {}^v\mathcal{A}$ . We generate  $N$ -anholonomic commutative geometries if we take  ${}^N\mathcal{A} \doteq C^\infty(\mathbf{V})$ , or  ${}^h\mathcal{A} \doteq C^\infty(h\mathbf{V})$ .

Let us show how it is possible to compute distance in a  $d$ -spinor space:

**Theorem 1.2.5** *Let  $({}^N\mathcal{A}, {}^N\mathcal{H}, \widehat{\mathbb{D}}, \mathbf{J}, {}_{[cr]}\Gamma)$  defines a noncommutative geometry being irreducible for  ${}^N\mathcal{A} \doteq C^\infty(\mathbf{V})$ , where  $\mathbf{V}$  is a compact, connected and oriented manifold without boundaries, of spectral dimension  $\dim \mathbf{V} = n + m$ . In this case, there are satisfied the conditions:*

1. *There is a unique  $\mathbf{g}(\widehat{\mathbb{D}}) = ({}^h g, {}^v g)$  with the "nonlinear" geodesic distance on  $\mathbf{V}$  defined by  $d(u_1, u_2) = \sup_{f \in C(\mathbf{V})} \left\{ f(u_1, u_2) / \|\widehat{\mathbb{D}}, f\| \leq 1 \right\}$ , for any smooth function  $f \in C(\mathbf{V})$ .*
2. *A  $N$ -anholonomic manifold  $\mathbf{V}$  is a spin  $N$ -anholonomic space, for which the operators  $\widehat{\mathbb{D}}'$  satisfying the condition  $\mathbf{g}(\widehat{\mathbb{D}}') = \mathbf{g}(\widehat{\mathbb{D}})$  define an union of affine spaces identified by the  $d$ -spinor structures on  $\mathbf{V}$ .*
3. *The functional  $\mathcal{S}(\widehat{\mathbb{D}}) \doteq \int |\widehat{\mathbb{D}}|^{-n-m+2}$  defines a quadratic  $d$ -form with  $(n + m)$ -splitting for every affine space which is minimal for  $\widehat{\mathbb{D}} = \overleftarrow{\widehat{\mathbb{D}}}$  as the canonical Dirac  $d$ -operator corresponding to the  $d$ -spin structure with the minimum proportional to the Einstein-Hilbert action constructed for the canonical  $d$ -connection with  $d$ -scalar curvature  ${}^s\mathbf{R}$ ,*

$$\mathcal{S}(\overleftarrow{\widehat{\mathbb{D}}}) = -\frac{n+m-2}{24} \int_{\mathbf{V}} {}^s\mathbf{R} \sqrt{{}^h g} \sqrt{{}^v h} dx^1 \dots dx^n \delta y^{n+1} \dots \delta y^{n+m}.$$

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<sup>22</sup>We omit in this paper the details on axiomatics and related proofs for such considerations.



The existence of a canonical  $d$ -connection structure which is metric compatible and constructed from the coefficients of the  $d$ -metric and  $N$ -connection structure is of crucial importance allowing the formulation and proofs of the main results of this work. As a matter of principle, we can consider any splitting of connections and compute a unique distance like we stated in the above Theorem 1.2.5, but for a "non-canonical" Dirac  $d$ -operator. This holds true for any noncommutative geometry induced by a metric compatible  $d$ -connection supposed to be uniquely induced by a metric tensor.

In more general cases, we can consider any metric compatible  $d$ -connection with arbitrary  $d$ -torsion. Such constructions can be also elaborated in  $N$ -adapted form by preserving the respective  $h$ - and  $v$ -irreducible decompositions. For the Dirac  $d$ -operators, we have to start with the Proposition 1.2.3 and then to repeat all constructions both on  $h$ - and  $v$ -subspaces. In this article, we do not analyze (non) commutative geometries enabled with general torsions but consider only nonholonomic deformations when distortions are induced by a metric structure.

Finally, we note that Theorem 1.2.5 allows us to extract from a canonical nonholonomic model of noncommutative geometry various types of commutative geometries (holonomic and  $N$ -anholonomic Riemannian spaces, Finsler-Lagrange spaces and generalizations) for corresponding nonholonomic Dirac operators.

#### 1.2.4 Noncommutative geometry and Ricci flows

The Ricci flow equations and Perelman functionals can be re-defined with respect to moving frames subjected to nonholonomic constraints.<sup>23</sup> Considering models of evolution of geometric objects in a form adapted to certain classes of nonholonomic constraints, we proved that metrics and connections defining (pseudo) Riemannian spaces may flow into similar non-holonomically deformed values modelling generalized Finsler and Lagrange configurations, with symmetric and nonsymmetric metrics, or possessing noncommutative symmetries.

The original Hamilton-Perelman constructions were for unconstrained

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<sup>23</sup>there are used also some other equivalent terms like anholonomic, or non-integrable, restrictions/ constraints; we emphasize that in classical and quantum physics the field and evolution equations play a fundamental role but together with certain types of constraints and broken symmetries; a rigorous mathematical approach to modern physical theories can be elaborated only following geometric methods from 'nonholonomic field theory and mechanics'

flows of metrics evolving only on (pseudo) Riemannian manifolds. There were proved a set of fundamental results in mathematics and physics (for instance, the Thurston and Poincaré conjectures, related to spacetime topological properties, Ricci flow running of physical constants and fields etc). Nevertheless, a number of important problems in geometry and physics are considered in the framework of classical and quantum field theories with constraints (for instance, the Lagrange and Hamilton mechanics, Dirac quantization of constrained systems, gauge theories with broken symmetries etc). With respect to the Ricci flow theory, to impose constraints on evolution equations is to extend the research programs on manifolds enabled with nonholonomic distributions, i.e. to study flows of fundamental geometric structures on nonholonomic manifolds.

Imposing certain noncommutative conditions on physical variables and coordinates in an evolution theory, we transfer the constructions and methods into the field of noncommutative geometric analysis on nonholonomic manifolds. This also leads naturally to various problems related to noncommutative generalizations of the Ricci flow theory and possible applications in modern physics. In this work, we follow the approach to noncommutative geometry when the spectral action paradigm, with spectral triples and Dirac operators, gives us a very elegant formulation of the standard model in physics.

Following the spectral action paradigm, all details of the standard models of particle interactions and gravity can be "extracted" from a noncommutative geometry generated by a spectral triple  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  by postulating the action

$$Tr \ f(\mathcal{D}^2/\Lambda^2) + \langle \Psi | \mathcal{D} | \Psi \rangle, \quad (1.29)$$

where "spectral" is in the sense that the action depends only on the spectrum of the Dirac operator  $\mathcal{D}$  on a certain noncommutative space defined by a noncommutative associative algebra  $\mathcal{A} = C^\infty(V) \otimes {}^P\mathcal{A}$ . In formula (1.29),  $Tr$  is the trace in operator algebra and  $\Psi$  is a spinor, all defined for a Hilbert space  $\mathcal{H}$ ,  $\Lambda$  is a cutoff scale and  $f$  is a positive function. For a number of physical applications,  ${}^P\mathcal{A}$  is a finite dimensional algebra and  $C^\infty(V)$  is the algebra of complex valued and smooth functions over a "space"  $V$ , a topological manifold, which for different purposes can be enabled with various necessary geometric structures. The spectral geometry of  $\mathcal{A}$  is given by the product rule  $\mathcal{H} = L^2(V, S) \otimes {}^P\mathcal{H}$ , where  $L^2(V, S)$  is the Hilbert space of  $L^2$  spinors and  ${}^P\mathcal{H}$  is the Hilbert space of quarks and leptons fixing the choice of the Dirac operator  ${}^PD$  and the action  ${}^P\mathcal{A}$  for fundamental particles. Usually, the Dirac operator from (1.29) is parametrized  $\mathcal{D} = {}^VD \otimes$

$1 + \gamma_5 \otimes {}^P D$ , where  ${}^V D$  is the Dirac operator of the Levi–Civita spin connection on  $V$ .<sup>24</sup>

In order to construct exact solutions with noncommutative symmetries and noncommutative gauge models of gravity and include dilaton fields, one has to use instead of  ${}^V D$  certain generalized types of Dirac operators defined by nonholonomic and/or conformal deformations of the ‘primary’ Levi–Civita spin connection. In a more general context, the problem of constructing well defined geometrically and physically motivated nonholonomic Dirac operators is related to the problem of definition of spinors and Dirac operators on Finsler–Lagrange spaces and generalizations.

### Spectral Functionals and Ricci Flows

The goal of this section is to prove that the Perelman’s functionals and their generalizations for nonholonomic Ricci flows in can be extracted from flows of a generalized Dirac operator  ${}^N \mathcal{D}(\chi) = \mathbb{D}(\chi) \otimes 1$  included in spectral functionals of type

$$Tr {}^b f({}^N \mathcal{D}^2(\chi)/\Lambda^2), \quad (1.30)$$

where  ${}^b f(\chi)$  are testing functions labelled by  $b = 1, 2, 3$  and depending on a real flow parameter  $\chi$ , which in the commutative variant of the Ricci flow theory corresponds to that for R. Hamilton’s equations. For simplicity, we shall use one cutoff parameter  $\Lambda$  and suppose that operators under flows act on the same algebra  $\mathcal{A}$  and Hilbert space  $\mathcal{H}$ , i.e. we consider families of spectral triples of type  $(\mathcal{A}, \mathcal{H}, {}^N \mathcal{D}(\chi))$ .<sup>25</sup>

**Definition 1.2.11** *The normalized Ricci flow equations (R. Hamilton’s equations) generalized on nonholonomic manifolds are defined in the form*

$$\frac{\partial \mathbf{g}_{\alpha\beta}(\chi)}{\partial \chi} = -2 {}^N \mathbf{R}_{\alpha\beta}(\chi) + \frac{2r}{5} \mathbf{g}_{\alpha\beta}(\chi), \quad (1.31)$$

where  $\mathbf{g}_{\alpha\beta}(\chi)$  defines a family of  $d$ -metrics parametrized in the form (1.91) on a  $N$ -anholonomic manifold  $\mathbf{V}$  enabled with a family of  $N$ -connections  $N_i^a(\chi)$ .

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<sup>24</sup>in this work, we shall use left “up” and “low” abstract labels which should not be considered as tensor or spinor indices written in the right side of symbols for geometrical objects

<sup>25</sup>we shall omit in this section the left label “N” for algebras and Hilbert spaces if that will not result in ambiguities

The effective "cosmological" constant  $2r/5$  in (1.31) with normalizing factor  $r = \int_v^N \mathbf{R} dv/v$  is introduced with the aim to preserve a volume  $v$  on  $\mathbf{V}$ , where  $\int_v^N \mathbf{R}$  is the scalar curvature.<sup>26</sup>

The corresponding family of Ricci tensors  ${}^N\mathbf{R}_{\alpha\beta}(\chi)$ , in (1.31), and non-holonomic Dirac operators  ${}^ND(\chi)$ , in (1.29), are defined for any value of  $\chi$  by a general metric compatible linear connection  ${}^N\mathbf{\Gamma}$  adapted to a N-connection structure. In a particular case, we can consider the Levi-Civita connection  $\Gamma$ , which is used in standard geometric approaches to physical theories. Nevertheless, for various purposes in modelling evolution of off-diagonal Einstein metrics, constrained physical systems, effective Finsler and Lagrange geometries, Fedosov quantization of field theories and gravity etc<sup>27</sup>, it is convenient to work with a "N-adapted" linear connection  ${}^N\mathbf{\Gamma}(\mathbf{g})$ . If such a connection is also uniquely defined by a metric structure  $\mathbf{g}$ , we are able to re-define the constructions in an equivalent form for the corresponding Levi-Civita connection.

In noncommutative geometry, all physical information on generalized Ricci flows can be encoded into a corresponding family of nonholonomic Dirac operators  ${}^ND(\chi)$ . For simplicity, in this work, we shall consider that  ${}^PD = 0$ , i.e. we shall not involve into the (non)commutative Ricci flow theory the particle physics. Perhaps a "comprehensive" noncommutative Ricci flow theory should include as a stationary case the "complete" spectral action (1.29) parametrized for the standard models of gravity and particle physics.

### Spectral flows and Perelman functionals

Let us consider a family of generalized d-operators

$$\mathcal{D}^2(\chi) = - \left\{ \frac{\mathbb{I}}{2} \mathbf{g}^{\alpha\beta}(\chi) [\mathbf{e}_\alpha(\chi)\mathbf{e}_\beta(\chi) + \mathbf{e}_\beta(\chi)\mathbf{e}_\alpha(\chi)] + \mathbf{A}^\nu(\chi)\mathbf{e}_\nu(\chi) + \mathbf{B}(\chi) \right\}, \quad (1.32)$$

where the real flow parameter  $\chi \in [0, \chi_0)$  and, for any fixed values of this parameter, the matrices  $\mathbf{A}^\nu(\chi)$  and  $\mathbf{B}(\chi)$  are determined by a N-anholonomic Dirac operator  $\mathbb{D}$  induced by a metric compatible d-connection  $\mathbf{D}$ , see and Definition 1.2.8; for the canonical d-connection, we have to put "hats" on

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<sup>26</sup>We note that in our work there used two mutually related flow parameters  $\chi$  and  $\tau$ ; for simplicity, in this work we write only  $\chi$  even, in general, such parameters should be rescaled for different geometric analysis constructions.

<sup>27</sup>the coefficients of corresponding N-connection structures being defined respectively by the generic off-diagonal metric terms, anholonomy frame coefficients, Finsler and Lagrange fundamental functions etc

symbols and write  $\widehat{\mathcal{D}}^2$ ,  $\widehat{\mathbf{A}}^\nu$  and  $\widehat{\mathbf{B}}$ . We introduce two functionals  $\mathcal{F}$  and  $\mathcal{W}$  depending on  $\chi$ ,

$$\mathcal{F} = Tr \left[ {}^1f(\chi)({}^1\phi \mathcal{D}^2(\chi)/\Lambda^2) \right] \simeq \sum_{k \geq 0} {}^1f_{(k)}(\chi) {}^1a_{(k)}({}^1\phi \mathcal{D}^2(\chi)/\Lambda^2) \quad (1.33)$$

$$\text{and } \mathcal{W} = {}^2\mathcal{W} + {}^3\mathcal{W}, \quad (1.34)$$

$$\text{for } {}^e\mathcal{W} = Tr \left[ {}^ef(\chi)({}^e\phi \mathcal{D}^2(\chi)/\Lambda^2) \right] = \sum_{k \geq 0} {}^ef_{(k)}(\chi) {}^ea_{(k)}({}^e\phi \mathcal{D}^2(\chi)/\Lambda^2),$$

where we consider a cutting parameter  $\Lambda^2$  for both cases  $e = 2, 3$ . Functions  ${}^bf$ , with label  $b$  taking values 1, 2, 3, have to be chosen in a form which insure that for a fixed  $\chi$  we get certain compatibility with gravity and particle physics and result in positive average energy and entropy for Ricci flows of geometrical objects. For such testing functions, ones hold true the formulas

$$\begin{aligned} {}^bf_{(0)}(\chi) &= \int_0^\infty {}^bf(\chi, u) u \, du, \quad {}^bf_{(2)}(\chi) = \int_0^\infty {}^bf(\chi, u) \, du, \\ {}^bf_{(2k+4)}(\chi) &= (-1)^k {}^bf^{(k)}(\chi, 0), \quad k \geq 0. \end{aligned} \quad (1.35)$$

We will comment the end of this subsection on dependence on  $\chi$  of such functions.

The coefficients  ${}^ba_{(k)}$  can be computed as the Seeley – de Witt coefficients (we chose such notations when in the holonomic case the scalar curvature is negative for spheres and the space is locally Euclidean). In functionals (1.33) and (1.34), we consider dynamical scaling factors of type  ${}^b\rho = \Lambda \exp({}^b\phi)$ , when, for instance,

$$\begin{aligned} {}^1\phi \mathcal{D}^2 &= e^{-{}^1\phi} \mathcal{D}^2 e^{{}^1\phi} \\ &= - \left\{ \frac{\mathbb{I}}{2} {}^1\phi \mathbf{g}^{\alpha\beta} \left[ {}^1\phi \mathbf{e}_\alpha {}^1\phi \mathbf{e}_\beta + {}^1\phi \mathbf{e}_\beta {}^1\phi \mathbf{e}_\alpha \right] + {}^1\phi \mathbf{A}^\nu {}^1\phi \mathbf{e}_\nu + {}^1\phi \mathbf{B} \right\}, \end{aligned} \quad (1.36)$$

$$\begin{aligned} \text{for } {}^1\phi \mathbf{A}^\nu &= e^{-2{}^1\phi} \times \mathbf{A}^\nu - 2 {}^1\phi \mathbf{g}^{\nu\mu} \times {}^1\phi \mathbf{e}_\beta ({}^1\phi), \\ {}^1\phi \mathbf{B} &= e^{-2{}^1\phi} \times \left( \mathbf{B} - \mathbf{A}^\nu {}^1\phi \mathbf{e}_\beta ({}^1\phi) \right) + {}^1\phi \mathbf{g}^{\nu\mu} \times {}^1\phi W_{\nu\mu}^\gamma {}^1\phi \mathbf{e}_\gamma, \end{aligned}$$

for re-scaled d-metric  ${}^1\phi \mathbf{g}_{\alpha\beta} = e^{2{}^1\phi} \times \mathbf{g}_{\alpha\beta}$  and N-adapted frames  ${}^1\phi \mathbf{e}_\alpha = e^{1\phi} \times \mathbf{e}_\alpha$  satisfying anholonomy relations, with re-scaled nonholonomy coefficients  ${}^1\phi W_{\nu\mu}^\gamma$ . We emphasize that similar formulas can be written by

substituting respectively the labels and scaling factors containing  ${}^1\phi$  with  ${}^2\phi$  and  ${}^3\phi$ . For simplicity, we shall omit left labels 1, 2, 3 for  $\phi$  and  $f, a$  if that will not result in ambiguities.

Let us denote by  ${}^s\mathbf{R}(\mathbf{g}_{\mu\nu})$  and  $\mathbf{C}_{\mu\nu\lambda\gamma}(\mathbf{g}_{\mu\nu})$ , correspondingly, the scalar curvature and conformal Weyl d-tensor <sup>28</sup>

$$\begin{aligned}\mathbf{C}_{\mu\nu\lambda\gamma} &= \mathbf{R}_{\mu\nu\lambda\gamma} + \frac{1}{2}(\mathbf{R}_{\mu\lambda}\mathbf{g}_{\nu\gamma} - \mathbf{R}_{\nu\lambda}\mathbf{g}_{\mu\gamma} - \mathbf{R}_{\mu\gamma}\mathbf{g}_{\nu\lambda} + \mathbf{R}_{\nu\gamma}\mathbf{g}_{\mu\lambda}) \\ &\quad - \frac{1}{6}(\mathbf{g}_{\mu\lambda}\mathbf{g}_{\nu\gamma} - \mathbf{g}_{\nu\lambda}\mathbf{g}_{\mu\gamma}) {}^s\mathbf{R},\end{aligned}$$

defined by a d-metric  $\mathbf{g}_{\mu\nu}$  and a metric compatible d-connection  $\mathbf{D}$  (in our approach,  $\mathbf{D}$  can be any d-connection constructed in a unique form from  $\mathbf{g}_{\mu\nu}$  and  $\mathbf{N}_i^a$  following a well defined geometric principle). For simplicity, we shall work on a four dimensional space and use values

$$\begin{aligned}\int d^4u \sqrt{\det |e^{2\phi}\mathbf{g}_{\mu\nu}|} \mathbf{R}(e^{2\phi}\mathbf{g}_{\mu\nu})^* \mathbf{R}^*(e^{2\phi}\mathbf{g}_{\mu\nu}) = \\ \int d^4u \sqrt{\det |\mathbf{g}_{\mu\nu}|} \mathbf{R}(\mathbf{g}_{\mu\nu})^* \mathbf{R}^*(\mathbf{g}_{\mu\nu}) = \frac{1}{4} \int d^4u \frac{\epsilon^{\mu\nu\alpha\beta}\epsilon_{\rho\sigma\gamma\delta}}{(\sqrt{\det |\mathbf{g}_{\mu\nu}|})} \mathbf{R}^{\rho\sigma}{}_{\mu\nu} \mathbf{R}^{\gamma\delta}{}_{\alpha\beta},\end{aligned}$$

for the curvature d-tensor  $\mathbf{R}^{\rho\sigma}{}_{\mu\nu}$ , where sub-integral values are defined by Chern-Gauss-Bonnet terms  $\mathbf{R}^* \mathbf{R}^* \equiv \frac{1}{4\sqrt{\det |\mathbf{g}_{\mu\nu}|}} \epsilon^{\mu\nu\alpha\beta}\epsilon_{\rho\sigma\gamma\delta} \mathbf{R}^{\rho\sigma}{}_{\mu\nu} \mathbf{R}^{\gamma\delta}{}_{\alpha\beta}$ .

One has the four dimensional approximation

$$\begin{aligned}Tr [f(\chi)(\phi\mathcal{D}^2(\chi)/\Lambda^2)] &\simeq \frac{45}{4\pi^2} f_{(0)} \int \delta^4u e^{2\phi} \sqrt{\det |\mathbf{g}_{\mu\nu}|} + \frac{15}{16\pi^2} \times (1.37) \\ f_{(2)} \int \delta^4u e^{2\phi} \sqrt{\det |\mathbf{g}_{\mu\nu}|} &({}^s\mathbf{R}(e^{2\phi}\mathbf{g}_{\mu\nu}) + 3e^{-2\phi}\mathbf{g}^{\alpha\beta}(\mathbf{e}_\alpha\phi \mathbf{e}_\beta\phi + \mathbf{e}_\beta\phi \mathbf{e}_\alpha\phi)) \\ + \frac{1}{128\pi^2} f_{(4)} \int \delta^4u e^{2\phi} \sqrt{\det |\mathbf{g}_{\mu\nu}|} &\times \\ (11 \mathbf{R}^*(e^{2\phi}\mathbf{g}_{\mu\nu}) \mathbf{R}^*(e^{2\phi}\mathbf{g}_{\mu\nu}) - 18 \mathbf{C}_{\mu\nu\lambda\gamma}(e^{2\phi}\mathbf{g}_{\mu\nu}) &\mathbf{C}^{\mu\nu\lambda\gamma}(e^{2\phi}\mathbf{g}_{\mu\nu})) .\end{aligned}$$

Let us state some additional hypotheses which will be used for proofs of the theorems in this section: Hereafter we shall consider a four dimensional compact N-anholonomic manifold  $\mathbf{V}$ , with volume forms  $\delta V = \sqrt{\det |\mathbf{g}_{\mu\nu}|} \delta^4u$  and normalization  $\int_{\mathbf{V}} \delta V \mu = 1$  for  $\mu = e^{-f}(4\pi\chi)^{-(n+m)/2}$  with  $f$  being a scalar function  $f(\chi, u)$  and  $\chi > 0$ .

Now, we are able to formulate the main results of this section:

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<sup>28</sup>for any metric compatible d-connection  $\mathbf{D}$ , the Weyl d-tensor can be computed by formulas similar to those for the Levi-Civita connection  $\nabla$ ; here we note that if a Weyl d-tensor is zero, in general, the Weyl tensor for  $\nabla$  does not vanish (and inversely)

**Theorem 1.2.6** *For the scaling factor  ${}^1\phi = -f/2$ , the spectral functional (1.33) can be approximated  $\mathcal{F} = {}^P\mathcal{F}(\mathbf{g}, \mathbf{D}, f)$ , where the first Perelman functional (in our case for  $N$ -anholonomic Ricci flows) is*

$${}^P\mathcal{F} = \int_{\mathbf{V}} \delta V e^{-f} \left[ {}_s\mathbf{R}(e^{-f} \mathbf{g}_{\mu\nu}) + \frac{3}{2} e^f \mathbf{g}^{\alpha\beta} (\mathbf{e}_\alpha f \mathbf{e}_\beta f + \mathbf{e}_\beta f \mathbf{e}_\alpha f) \right].$$

There are some important remarks.

**Remark 1.2.1** *For nonholonomic Ricci flows of (non)commutative geometries, we have to adapt the evolution to certain  $N$ -connection structures (i.e. nonholonomic constraints). This results in additional possibilities to re-scale coefficients and parameters in spectral functionals and their commutative limits:*

1. *The evolution parameter  $\chi$ , scaling factors  ${}^b f$  and nonholonomic constraints and coordinates can be re-scaled/ redefined (for instance,  $\chi \rightarrow \check{\chi}$  and  ${}^b f \rightarrow {}^b \check{f}$ ) such a way that the spectral functionals have limits to some 'standard' nonholonomic versions of Perelman functionals (with prescribed types of coefficients).*
2. *Using additional dependencies on  $\chi$  and freedom in choosing scaling factors  ${}^b f(\chi)$ , we can prescribe such nonholonomic constraints/ configurations on evolution equations (for instance, with  ${}^1\check{f}_{(2)} = 16\pi^2/15$  and  ${}^1\check{f}_{(0)} = {}^1\check{f}_{(4)} = 0$ ) when the spectral functionals result exactly in necessary types of effective Perelman functionals (with are commutative, but, in general, nonholonomic).*
3. *For simplicity, we shall write in brief only  $\chi$  and  $f$  considering that we have chosen such scales, parametrizations of coordinates and  $N$ -adapted frames and flow parameters when coefficients in spectral functionals and resulting evolution equations maximally correspond to certain generally accepted commutative physical actions/ functionals.*
4. *For nonholonomic Ricci flow models (commutative or noncommutative ones) with a fixed evolution parameter  $\chi$ , we can construct certain effective nonholonomic evolution models with induced noncommutative corrections for coefficients.*
5. *Deriving effective nonholonomic evolution models from spectral functionals, we can use the technique of "extracting" physical models from spectral actions. For commutative and/or noncommutative geometric/*

*physical models of nonholonomic Ricci flows, we have to generalize the approach to include spectral functionals and  $N$ -adapted evolution equations depending on the type of nonholonomic constraints, normalizations and re-scalings of constants and effective conformal factors.*

We "extract" from the second spectral functional (1.34) another very important physical value:

**Theorem 1.2.7** *The functional (1.34) is approximated  $\mathcal{W} = {}^P\mathcal{W}(\mathbf{g}, \mathbf{D}, f, \chi)$ , where the second Perelman functional is*

$${}^P\mathcal{W} = \int_V \delta V \mu \times [\chi \left( {}_s\mathbf{R}(e^{-f} \mathbf{g}_{\mu\nu}) + \frac{3}{2} e^f \mathbf{g}^{\alpha\beta} (\mathbf{e}_\alpha f \mathbf{e}_\beta f + \mathbf{e}_\beta f \mathbf{e}_\alpha f) \right) + f - (n+m)],$$

*for scaling  ${}^2\phi = -f/2$  in  ${}^2\mathcal{W}$  and  ${}^3\phi = (\ln |f - (n+m)| - f)/2$  in  ${}^3\mathcal{W}$ , from (1.34).*

The nonholonomic version of Hamilton equations (1.31) can be derived from commutative Perelman functionals  ${}^P\mathcal{F}$  and  ${}^P\mathcal{W}$ . The original Hamilton–Perelman Ricci flows constructions can be generated for  $\mathbf{D} = \nabla$ . The surprising result is that even we start with a Levi–Civita linear connection, the nonholonomic evolution will result almost sure in generalized geometric configurations with various  $\mathbf{N}$  and  $\mathbf{D}$  structures.

### Spectral functionals for thermodynamical values

Certain important thermodynamical values such as the average energy and entropy can be derived directly from noncommutative spectral functionals as respective commutative configurations of spectral functionals of type (1.33) and (1.34) but with different testing functions than in Theorems 1.2.6 and 1.2.7.

**Theorem 1.2.8** *Using a scaling factor of type  ${}^1\phi = -f/2$ , we extract from the spectral functional (1.33) a nonholonomic version of average energy,  $\mathcal{F} \rightarrow \langle \mathcal{E} \rangle$ , where*

$$\langle \mathcal{E} \rangle = -\chi^2 \int_V \delta V \mu \left[ {}_s\mathbf{R}(e^{-f} \mathbf{g}_{\mu\nu}) + \frac{3}{2} \mathbf{g}^{\alpha\beta} (\mathbf{e}_\alpha f \mathbf{e}_\beta f + \mathbf{e}_\beta f \mathbf{e}_\alpha f) - \frac{n+m}{2\chi} \right] \quad (1.38)$$

*if the testing function is chosen to satisfy the conditions  ${}^1f_{(0)}(\chi) = 4\pi^2(n+m)\chi/45(4\pi\chi)^{(n+m)/2}$ ,  ${}^1f_{(2)}(\chi) = 16\pi^2\chi^2/15(4\pi\chi)^{(n+m)/2}$  and  ${}^1f_{(4)}(\chi) = 0$ .*



Similarly to Theorem 1.2.7 (inverting the sign of nontrivial coefficients of the testing function) we prove:

**Theorem 1.2.9** *We extract a nonholonomic version of entropy of nonholonomic Ricci flows from the functional (1.34),  $\mathcal{W} \rightarrow \mathcal{S}$ , where*

$$\mathcal{S} = - \int_{\mathbf{V}} \delta V \mu [\chi \left( {}_s\mathbf{R}(e^{-f} \mathbf{g}_{\mu\nu}) - \frac{3}{2} e^f \mathbf{g}^{\alpha\beta} (\mathbf{e}_\alpha f \mathbf{e}_\beta f + \mathbf{e}_\beta f \mathbf{e}_\alpha f) \right) + f - (n+m)],$$

*if we introduce  $\delta V = \delta^4 u$  and  $\mu = e^{-f} (4\pi\chi)^{-(n+m)/2}$  into formula (1.37), for  $\chi > 0$  and  $\int_{\mathbf{V}} dV \mu = 1$  in (1.37), for scaling  ${}^2\phi = -f/2$  in  ${}^2\mathcal{W}$  and  ${}^3\phi = (\ln |f - (n+m)| - f)/2$  in  ${}^3\mathcal{W}$ , from (1.34).*

We can formulate and prove a Theorem alternative to Theorem 1.2.8 and get the formula (1.38) from the spectral functional  ${}^2\mathcal{W} + {}^3\mathcal{W}$ . Such a proof is similar to that for Theorem 1.2.7, but with corresponding nontrivial coefficients for two testing functions  ${}^2f(\chi)$  and  ${}^3f(\chi)$ . The main difference is that for Theorem 1.2.8 it is enough to use only one testing function. We do not present such computations in this work.

It is not surprising that certain 'commutative' thermodynamical physical values can be derived alternatively from different spectral functionals because such type 'commutative' thermodynamical values can be generated by a partition function

$$\widehat{Z} = \exp \left\{ \int_{\mathbf{V}} \delta V \mu \left[ -f + \frac{n+m}{2} \right] \right\}, \quad (1.39)$$

associated to any  $Z = \int \exp(-\beta E) d\omega(E)$  being the partition function for a canonical ensemble at temperature  $\beta^{-1}$ , which in its turn is defined by the measure taken to be the density of states  $\omega(E)$ . In this case, we can compute the average energy,  $\langle E \rangle = -\partial \log Z / \partial \beta$ , the entropy  $S = \beta \langle E \rangle + \log Z$  and the fluctuation  $\sigma = \langle (E - \langle E \rangle)^2 \rangle = \partial^2 \log Z / \partial \beta^2$ .

**Remark 1.2.2** *Following a straightforward computation for (1.39) we prove that*

$$\widehat{\sigma} = 2\chi^2 \int_{\mathbf{V}} \delta V \mu \left[ \left| R_{ij} + D_i D_j f - \frac{1}{2\chi} g_{ij} \right|^2 + \left| R_{ab} + D_a D_b f - \frac{1}{2\chi} g_{ab} \right|^2 \right]. \quad (1.40)$$

Using formula  $\mathbf{R}_{\mu\nu}^2 = \frac{1}{2} \mathbf{C}_{\mu\nu\rho\sigma}^2 - \frac{1}{2} \mathbf{R}^* \mathbf{R}^* + \frac{1}{3} {}_s\mathbf{R}^2$  (it holds true for any metric compatible d-connections, similarly to the formula for the Levi-Civita connection, we expect that the formula for fluctuations (1.40) can

be generated directly, by corresponding re-scalings, from a spectral action with nontrivial coefficients for testing functions when  $f_{(4)} \neq 0$ , see formula (1.37). Here we note that in the original Perelman's functionals there were not introduced terms being quadratic on curvature/ Weyl / Ricci tensors. For nonzero  $f_{(4)}$ , such terms may be treated as certain noncommutative / quantum contributions to the classical commutative Ricci flow theory. For simplicity, we omit such considerations in this work.

The framework of Perelman's functionals and generalizations to corresponding spectral functionals can be positively applied for developing statistical analogies of (non) commutative Ricci flows. For instance, the functional  $\mathcal{W}$  is the "opposite sign" entropy, see formulas from Theorems 1.2.7 and 1.2.9. Such constructions may be considered for a study of optimal "topological" configurations and evolution of both commutative and non-commutative geometries and relevant theories of physical interactions.

### 1.2.5 (Non) commutative gauge gravity

We consider main results of Refs. [1, 30] concerning noncommutative gauge models of gravity:

The basic idea was to use a geometrical result due to D. A. Popov and I. I. Dikhhin (1976) that the Einstein gravity can be equivalently represented as a gauge theory with a Cartan type connection in the bundle of affine frames. Such gauge theories are with nonsemisimple structure gauge groups, i. e. with degenerated metrics in the total spaces. Using an auxiliary symmetric form for the typical fiber, any such model can be transformed into a variational one. There is an alternative way to construct geometrically a usual Yang–Mills theory by applying a corresponding set of absolute derivations and dualities defined by the Hodge operator. For both approaches, there is a projection formalism reducing the geometric field equations on the base space to be exactly the Einstein equations from the general relativity theory.

For more general purposes, it was suggested to consider also extensions to a nonlinear realization with the (anti) de Sitter gauge structural group (A. Tseytlin, 1982). The constructions with nonlinear group realizations are very important because they prescribe a consistent approach of distinguishing the frame indices and coordinate indices subjected to different rules of transformation. This approach to gauge gravity (of course, after a corresponding generalizations of the Seiberg–Witten map) may include, in general, quadratic on curvature and torsion terms.

## Nonlinear gauge models for the (anti) de Sitter group

We introduce vielbein decompositions of (in general) complex metrics

$$\widehat{g}_{\alpha\beta}(u) = e_{\alpha}^{\alpha'}(u) e_{\beta}^{\beta'}(u) \eta_{\alpha'\beta'}, \quad e_{\alpha}^{\alpha'} e_{\alpha'}^{\beta} = \delta_{\alpha}^{\beta} \quad \text{and} \quad e_{\alpha}^{\alpha'} e_{\beta'}^{\alpha} = \delta_{\beta'}^{\alpha'},$$

where  $\eta_{\alpha'\beta'}$  is a constant diagonal matrix (for real spacetimes we can consider it as the flat Minkowski metric, for instance,  $\eta_{\alpha'\beta'} = \text{diag}(-1, +1, \dots, +1)$ ) and  $\delta_{\alpha}^{\beta}$  and  $\delta_{\beta'}^{\alpha'}$  are Kronecker's delta symbols. The vielbeins with an associated N-connection structure  $N_i^a(x^j, y^a)$ , being real or complex valued functions, have a special parametrization

$$e_{\alpha}^{\alpha'}(u) = \begin{bmatrix} e_i^{i'}(x^j) & N_i^c(x^j, y^a) & e_c^{b'}(x^j, y^a) \\ 0 & e_e^{e'}(x^j, y^a) & \end{bmatrix} \quad (1.41)$$

and

$$e_{\alpha'}^{\alpha}(u) = \begin{bmatrix} e^{i}_{i'}(x^j) & -N_i^c(x^j, y^a) & e^{i}_{i'}(x^j) \\ 0 & e^c_{c'}(x^j, y^a) & \end{bmatrix} \quad (1.42)$$

with  $e_i^{i'}(x^j)$  and  $e_c^{b'}(x^j, y^a)$  generating the coefficients of a metric defined with respect to anholonomic frames,

$$g_{ij}(x^j) = e_i^{i'}(x^j) e_j^{j'}(x^j) \eta_{i'j'}, \quad \text{and} \quad h_{ab}(x^j, y^c) = e_a^{a'}(x^j, y^c) e_b^{b'}(x^j, y^c) \eta_{a'b'}. \quad (1.43)$$

By using vielbeins and metrics of type (1.41) and (1.42) and, respectively, (1.43), we can model in a unified manner various types of (pseudo) Riemannian, Einstein–Cartan, Riemann–Finsler and vector/ covector bundle nonlinear connection commutative and noncommutative geometries in effective gauge and string theories (it depends on the parametrization of  $e_i^{i'}$ ,  $e_c^{b'}$  and  $N_i^c$  on coordinates and anholonomy relations).

We consider the de Sitter space  $\Sigma^4$  as a hypersurface defined by the equations  $\eta_{AB} u^A u^B = -l^2$  in the four dimensional flat space enabled with diagonal metric  $\eta_{AB}, \eta_{AA} = \pm 1$  (in this section  $A, B, C, \dots = 1, 2, \dots, 5$ ), where  $\{u^A\}$  are global Cartesian coordinates in  $\mathbb{R}^5$ ;  $l > 0$  is the curvature of de Sitter space (for simplicity, we consider here only the de Sitter case; the anti-de Sitter configuration is to be stated by a hypersurface  $\eta_{AB} u^A u^B = l^2$ ). The de Sitter group  $S_{(\eta)} = SO_{(\eta)}(5)$  is the isometry group of  $\Sigma^5$ -space with 6 generators of Lie algebra  $so_{(\eta)}(5)$  satisfying the commutation relations

$$[M_{AB}, M_{CD}] = \eta_{AC} M_{BD} - \eta_{BC} M_{AD} - \eta_{AD} M_{BC} + \eta_{BD} M_{AC}. \quad (1.44)$$

We can decompose the capital indices  $A, B, \dots$  as  $A = (\alpha', 5), B = (\beta', 5), \dots$ , and the metric  $\eta_{AB}$  as  $\eta_{AB} = (\eta_{\alpha'\beta'}, \eta_{55})$ . The operators (1.44)

$M_{AB}$  can be decomposed as  $M_{\alpha'\beta'} = \mathcal{F}_{\alpha'\beta'}$  and  $P_{\alpha'} = l^{-1}M_{5\alpha'}$  written as

$$\begin{aligned} [\mathcal{F}_{\alpha'\beta'}, \mathcal{F}_{\gamma'\delta'}] &= \eta_{\alpha'\gamma'} \mathcal{F}_{\beta'\delta'} - \eta_{\beta'\gamma'} \mathcal{F}_{\alpha'\delta'} + \eta_{\beta'\delta'} \mathcal{F}_{\alpha'\gamma'} - \eta_{\alpha'\delta'} \mathcal{F}_{\beta'\gamma'}, \\ [P_{\alpha'}, P_{\beta'}] &= -l^{-2} \mathcal{F}_{\alpha'\beta'}, \quad [P_{\alpha'}, \mathcal{F}_{\beta'\gamma'}] = \eta_{\alpha'\beta'} P_{\underline{\gamma}} - \eta_{\alpha'\gamma'} P_{\beta'}, \end{aligned} \quad (1.45)$$

where the Lie algebra  $so_{(\eta)}(5)$  is split into a direct sum,  $so_{(\eta)}(5) = so_{(\eta)}(4) \oplus V_4$  with  $V_4$  being the vector space stretched on vectors  $P_{\underline{\alpha}}$ . We remark that  $\Sigma^4 = S_{(\eta)}/L_{(\eta)}$ , where  $L_{(\eta)} = SO_{(\eta)}(4)$ . For  $\eta_{AB} = \text{diag}(-1, +1, +1, +1)$  and  $S_{10} = SO(1, 4)$ ,  $L_6 = SO(1, 3)$  is the group of Lorentz rotations.

The generators  $I^{\underline{a}}$  and structure constants  $f_{\underline{l}}^{\underline{sp}}$  of the de Sitter Lie group can be parametrized in a form distinguishing the de Sitter generators and commutations (1.45). The action of the group  $S_{(\eta)}$  may be realized by using  $4 \times 4$  matrices with a parametrization distinguishing the subgroup  $L_{(\eta)}$ :  $B = bB_L$ , where  $B_L = \begin{pmatrix} L & 0 \\ 0 & 1 \end{pmatrix}$ ,  $L \in L_{(\eta)}$  is the de Sitter boost matrix transforming the vector  $(0, 0, \dots, \rho) \in \mathbb{R}^5$  into the arbitrary point  $(V^1, V^2, \dots, V^5) \in \Sigma_{\rho}^5 \subset \mathcal{R}^5$  with curvature  $\rho$ ,  $(V_A V^A = -\rho^2, V^A = \tau^A \rho)$ , and the matrix  $b$  is expressed  $b = \begin{pmatrix} \delta^{\alpha'}_{\beta'} + \frac{\tau^{\alpha'} \tau_{\beta'}}{(1+\tau^5)} & \tau^{\alpha'} \\ \tau_{\beta'} & \tau^5 \end{pmatrix}$ . The de Sitter gauge field is associated with a  $so_{(\eta)}(5)$ -valued connection 1-form

$$\tilde{\Omega} = \begin{pmatrix} \omega^{\alpha'}_{\beta'} & \tilde{\theta}^{\alpha'} \\ \tilde{\theta}_{\beta'} & 0 \end{pmatrix}, \quad (1.46)$$

where  $\omega^{\alpha'}_{\beta'} \in so(4)_{(\eta)}$ ,  $\tilde{\theta}^{\alpha'} \in \mathcal{R}^4$ ,  $\tilde{\theta}_{\beta'} \in \eta_{\beta'\alpha'} \tilde{\theta}^{\alpha'}$ .

The actions of  $S_{(\eta)}$  mix the components of the matrix  $\omega^{\alpha'}_{\beta'}$  and  $\tilde{\theta}^{\alpha'}$  fields in (1.46). Because the introduced parametrization is invariant on action on  $SO_{(\eta)}(4)$  group, we cannot identify  $\omega^{\alpha'}_{\beta'}$  and  $\tilde{\theta}^{\alpha'}$ , respectively, with the connection  $\Gamma^{[c]}$  and the 1-form  $e^{\alpha}$  defined by a N-connection structure with the coefficients chosen as in (1.41) and (1.42). To avoid this difficulty we can consider nonlinear gauge realizations of the de Sitter group  $S_{(\eta)}$  by introducing the nonlinear gauge field

$$\Gamma = b^{-1} \tilde{\Omega} b + b^{-1} db = \begin{pmatrix} \Gamma^{\alpha'}_{\beta'} & \theta^{\alpha'} \\ \theta_{\beta'} & 0 \end{pmatrix}, \quad (1.47)$$

$$\begin{aligned} \text{where } \Gamma^{\alpha'}_{\beta'} &= \omega^{\alpha'}_{\beta'} - \left( \tau^{\alpha'} D\tau_{\beta'} - \tau_{\beta'} D\tau^{\alpha'} \right) / (1 + \tau^5), \\ \theta^{\alpha'} &= \tau^5 \tilde{\theta}^{\alpha'} + D\tau^{\alpha'} - \tau^{\alpha'} \left( d\tau^5 + \tilde{\theta}_{\gamma'} \tau^{\gamma'} \right) / (1 + \tau^5), \\ D\tau^{\alpha'} &= d\tau^{\alpha'} + \omega^{\alpha'}_{\beta'} \tau^{\beta'}. \end{aligned}$$

The action of the group  $S(\eta)$  is nonlinear, yielding the transformation rules  $\Gamma' = L'\Gamma(L')^{-1} + L'd(L')^{-1}$ ,  $\theta' = L\theta$ , where the nonlinear matrix-valued function  $L' = L'(\tau^\alpha, b, B_T)$  is defined from  $B_b = b'B_{L'}$ . The de Sitter 'nonlinear' algebra is defined by generators (1.45) and nonlinear gauge transforms of type (1.47).

### De Sitter Nonlinear Gauge Gravity and General Relativity

We generalize the constructions from Refs [1] to the case when the de Sitter nonlinear gauge gravitational connection (1.47) is defined by the viebeins (1.41) and (1.42) and the linear connection  $\Gamma^{[c]\alpha}_{\beta\mu} = \{\Gamma^\alpha_{\beta\mu}\}$ ,

$$\Gamma = \begin{pmatrix} \Gamma^{\alpha'}_{\beta'} & l_0^{-1}e^{\alpha'} \\ l_0^{-1}e_{\beta'} & 0 \end{pmatrix} \quad (1.48)$$

where

$$\begin{aligned} \Gamma^{\alpha'}_{\beta'} &= \Gamma^{\alpha'}_{\beta'\mu}\delta u^\mu, \\ \text{for } \Gamma^{\alpha'}_{\beta'\mu} &= e_\alpha^{\alpha'}e^\beta_{\beta'}\Gamma^\alpha_{\beta\mu} + e_\alpha^{\alpha'}\delta_\mu^{\beta'}e^\alpha_{\beta'}, \quad e^{\alpha'} = e_\mu^{\alpha'}\delta u^\mu, \end{aligned} \quad (1.49)$$

and  $l_0$  being a dimensional constant.

The matrix components of the curvature of the connection (1.48),

$$\mathcal{R}^{(\Gamma)} = d\Gamma + \Gamma \wedge \Gamma,$$

can be written

$$\mathcal{R}^{(\Gamma)} = \begin{pmatrix} \mathcal{R}^{\alpha'}_{\beta'} + l_0^{-1}\pi^{\alpha'}_{\beta'} & l_0^{-1}T^{\alpha'} \\ l_0^{-1}T^{\beta'} & 0 \end{pmatrix}, \quad (1.50)$$

for  $\pi^{\alpha'}_{\beta'} = e^{\alpha'} \wedge e_{\beta'}$ ,  $\mathcal{R}^{\alpha'}_{\beta'} = \frac{1}{2}\mathcal{R}^{\alpha'}_{\beta'\mu\nu}\delta u^\mu \wedge \delta u^\nu$ ,  $\mathcal{R}^{\alpha'}_{\beta'\mu\nu} = e^\beta_{\beta'}e_\alpha^{\alpha'}R^\alpha_{\beta\mu\nu}$ , with the coefficients  $R^\alpha_{\beta\mu\nu}$  defined with h-v-invariant components.

The de Sitter gauge group is semisimple: we are able to construct a variational gauge gravitational theory with the Lagrangian

$$L = L_{(g)} + L_{(m)} \quad (1.51)$$

where the gauge gravitational Lagrangian is defined

$$L_{(g)} = \frac{1}{4\pi}Tr\left(\mathcal{R}^{(\Gamma)} \wedge *_G\mathcal{R}^{(\Gamma)}\right) = \mathcal{L}_{(G)}|g|^{1/2}\delta^4u,$$

for  $\mathcal{L}_{(g)} = \frac{1}{2l^2} T^{\alpha'}_{\mu\nu} T^{\mu\nu}_{\alpha'} + \frac{1}{8\lambda} \mathcal{R}^{\alpha'}_{\beta'\mu\nu} \mathcal{R}^{\beta'\mu\nu}_{\alpha'} - \frac{1}{l^2} \left( \overleftarrow{R}(\Gamma) - 2\lambda_1 \right)$ , with  $\delta^4 u$  being the volume element,  $|g|$  is the determinant computed the metric coefficients stated with respect to N-elongated frames,  $T^{\alpha'}_{\mu\nu} = e^{\alpha'}_{\alpha} T^{\alpha}_{\mu\nu}$  (the gravitational constant  $l^2$  satisfies the relations  $l^2 = 2l_0^2 \lambda$ ,  $\lambda_1 = -3/l_0$ ),  $Tr$  denotes the trace on  $\alpha', \beta'$  indices. The matter field Lagrangian from (1.51) is defined

$$L_{(m)} = -\frac{1}{2} Tr(\Gamma \wedge *_g \mathcal{I}) = \mathcal{L}_{(m)} |g|^{1/2} \delta^n u,$$

with the Hodge operator derived by  $|g|$  and  $|h|$  where

$$\mathcal{L}_{(m)} = \frac{1}{2} \Gamma^{\alpha'}_{\beta'\mu} S^{\beta'\mu}_{\alpha} - t^{\mu}_{\alpha} l^{\alpha'}_{\mu}.$$

The matter field source  $\mathcal{J}$  is obtained as a variational derivation of  $\mathcal{L}_{(m)}$  on  $\Gamma$  and is parametrized in the form  $\mathcal{J} = \begin{pmatrix} S^{\alpha'}_{\beta} & -l_0 \tau^{\alpha'} \\ -l_0 \tau_{\beta'} & 0 \end{pmatrix}$ , with  $\tau^{\alpha'} = \tau^{\alpha'}_{\mu} \delta u^{\mu}$  and  $S^{\alpha'}_{\beta'} = S^{\alpha'}_{\beta'\mu} \delta u^{\mu}$  being respectively the canonical tensors of energy-momentum and spin density.

Varying the action  $S = \int \delta^4 u (\mathcal{L}_{(g)} + \mathcal{L}_{(m)})$  on the  $\Gamma$ -variables (1.48), we obtain the gauge-gravitational field equations:

$$d\left(*\mathcal{R}^{(\Gamma)}\right) + \Gamma \wedge \left(*\mathcal{R}^{(\Gamma)}\right) - \left(*\mathcal{R}^{(\Gamma)}\right) \wedge \Gamma = -\lambda(*\mathcal{J}), \quad (1.52)$$

where the Hodge operator  $*$  is used. This equations can be alternatively derived in geometric form by applying the absolute derivation and dual operators.

Distinguishing the variations on  $\Gamma$  and  $e$ -variables, we rewrite (1.52)

$$\begin{aligned} \widehat{D}\left(*\mathcal{R}^{(\Gamma)}\right) + \frac{2\lambda}{l^2} (\widehat{D}(*\pi) + e \wedge (*T^T) - (*T) \wedge e^T) &= -\lambda(*S), \\ \widehat{D}(*T) - \left(*\mathcal{R}^{(\Gamma)}\right) \wedge e - \frac{2\lambda}{l^2} (*\pi) \wedge e &= \frac{l^2}{2} \left(*t + \frac{1}{\lambda} * \varsigma\right), \end{aligned}$$

$e^T$  being the transposition of  $e$ , where

$$\begin{aligned} T^t &= \{T_{\alpha'} = \eta_{\alpha'\beta'} T^{\beta'}, T^{\beta'} = \frac{1}{2} T^{\beta'}_{\mu\nu} \delta u^{\mu} \wedge \delta u^{\nu}\}, \\ e^T &= \{e_{\alpha'} = \eta_{\alpha'\beta'} e^{\beta'}, e^{\beta'} = e^{\beta'}_{\mu} \delta u^{\mu}\}, \quad \widehat{D} = \delta + \widehat{\Gamma}, \end{aligned}$$

( $\widehat{\Gamma}$  acts as  $\Gamma^{\alpha'}_{\beta'\mu}$  on indices  $\gamma', \delta', \dots$  and as  $\Gamma^{\alpha}_{\beta\mu}$  on indices  $\gamma, \delta, \dots$ ). The value  $\varsigma$  defines the energy-momentum tensor of the gauge gravitational field  $\widehat{\Gamma} : \varsigma_{\mu\nu} \left(\widehat{\Gamma}\right) = \frac{1}{2} Tr(\mathcal{R}_{\mu\alpha} \mathcal{R}^{\alpha}_{\nu} - \frac{1}{4} \mathcal{R}_{\alpha\beta} \mathcal{R}^{\alpha\beta} G_{\mu\nu})$ .

Equations (1.52) make up the complete system of variational field equations for the nonlinear de Sitter gauge gravity. We note that we can obtain a nonvariational Poincaré gauge gravitational theory if we consider the contraction of the gauge potential (1.48) to a potential  $\Gamma^{[P]}$  with values in the Poincaré Lie algebra

$$\Gamma = \begin{pmatrix} \Gamma^{\alpha'}_{\beta'} & l_0^{-1}e^{\alpha'} \\ l_0^{-1}e_{\beta'} & 0 \end{pmatrix} \rightarrow \Gamma^{[P]} = \begin{pmatrix} \Gamma^{\alpha'}_{\beta'} & l_0^{-1}e^{\alpha'} \\ 0 & 0 \end{pmatrix}. \quad (1.53)$$

A similar gauge potential was considered in the formalism of linear and affine frame bundles on curved spacetimes by D. Popov and I. Dikhin. They considered the gauge potential (1.53) to be just the Cartan connection form in the affine gauge like gravity and proved that the Yang–Mills equations of their theory are equivalent, after projection on the base, to the Einstein equations.

### Enveloping algebras for gauge gravity connections

We define the gauge fields on a noncommutative space as elements of an algebra  $\mathcal{A}_u$  that form a representation of the generator  $I$ -algebra for the de Sitter gauge group and the noncommutative space is modelled as the associative algebra of  $\mathbf{IC}$ . This algebra is freely generated by the coordinates modulo ideal  $\mathcal{R}$  generated by the relations (one accepts formal power series)  $\mathcal{A}_u = \mathbf{IC}[[\hat{u}^1, \dots, \hat{u}^N]]/\mathcal{R}$ . A variational gauge gravitational theory can be formulated by using a minimal extension of the affine structural group  $\mathcal{A}f_{3+1}(\mathbb{R})$  to the de Sitter gauge group  $S_{10} = SO(4+1)$  acting on  $\mathbb{R}^{4+1}$ .

The gauge fields are elements of the algebra  $\hat{\psi} \in \mathcal{A}_I^{(dS)}$  that form the nonlinear representation of the de Sitter algebra  $so_{(\eta)}(5)$  (the whole algebra is denoted  $\mathcal{A}_z^{(dS)}$ ). The elements transform  $\delta\hat{\psi} = i\hat{\gamma}\hat{\psi}, \hat{\psi} \in \mathcal{A}_u, \hat{\gamma} \in \mathcal{A}_z^{(dS)}$ , under a nonlinear de Sitter transformation. The action of the generators (1.45) on  $\hat{\psi}$  is defined as the resulting element will form a nonlinear representation of  $\mathcal{A}_I^{(dS)}$  and, in consequence,  $\delta\hat{\psi} \in \mathcal{A}_u$  despite  $\hat{\gamma} \in \mathcal{A}_z^{(dS)}$ . We emphasize that for any representation the object  $\hat{\gamma}$  takes values in enveloping de Sitter algebra but not in a Lie algebra as would be for commuting spaces. We introduce a connection  $\hat{\Gamma}^\nu \in \mathcal{A}_z^{(dS)}$  in order to define covariant coordinates,  $\hat{U}^\nu = \hat{u}^\nu + \hat{\Gamma}^\nu$ . The values  $\hat{U}^\nu\hat{\psi}$  transform covariantly, i. e.  $\delta\hat{U}^\nu\hat{\psi} = i\hat{\gamma}\hat{U}^\nu\hat{\psi}$ , if and only if the connection  $\hat{\Gamma}^\nu$  satisfies the transformation law of the enveloping nonlinear realized de Sitter algebra,  $\delta\hat{\Gamma}^\nu\hat{\psi} = -i[\hat{u}^\nu, \hat{\gamma}] + i[\hat{\gamma}, \hat{\Gamma}^\nu]$ , where  $\delta\hat{\Gamma}^\nu \in \mathcal{A}_z^{(dS)}$ .

The enveloping algebra-valued connection has infinitely many component fields. Nevertheless, all component fields can be induced from a Lie algebra-valued connection by a Seiberg–Witten map for  $SO(n)$  and  $Sp(n)$ . Here, we show that similar constructions can be performed for nonlinear realizations of de Sitter algebra when the transformation of the connection is considered  $\delta\hat{\Gamma}^\nu = -i[u^\nu, * \hat{\gamma}] + i[\hat{\gamma}, * \hat{\Gamma}^\nu]$ . We treat in more detail the canonical case with the star product. The first term in the variation  $\delta\hat{\Gamma}^\nu$  gives  $-i[u^\nu, * \hat{\gamma}] = \theta^{\nu\mu} \frac{\partial}{\partial u^\mu} \gamma$ . Assuming that the variation of  $\hat{\Gamma}^\nu = \theta^{\nu\mu} Q_\mu$  starts with a linear term in  $\theta$ , we have  $\delta\hat{\Gamma}^\nu = \theta^{\nu\mu} \delta Q_\mu$ ,  $\delta Q_\mu = \frac{\partial}{\partial u^\mu} \gamma + i[\hat{\gamma}, * Q_\mu]$ . We expand the star product in  $\theta$  but not in  $g_a$  and find up to first order in  $\theta$  that

$$\gamma = \gamma_{\underline{a}}^1 I^{\underline{a}} + \gamma_{\underline{ab}}^1 I^{\underline{a}} I^{\underline{b}} + \dots, Q_\mu = q_{\mu, \underline{a}}^1 I^{\underline{a}} + q_{\mu, \underline{ab}}^2 I^{\underline{a}} I^{\underline{b}} + \dots \quad (1.54)$$

where  $\gamma_{\underline{a}}^1$  and  $q_{\mu, \underline{a}}^1$  are of order zero in  $\theta$  and  $\gamma_{\underline{ab}}^1$  and  $q_{\mu, \underline{ab}}^2$  are of second order in  $\theta$ . The expansion in  $I^{\underline{b}}$  leads to an expansion in  $g_a$  of the  $*$ -product because the higher order  $I^{\underline{b}}$ -derivatives vanish. For de Sitter case, we take the generators  $I^{\underline{b}}$  (1.45), with the corresponding de Sitter structure constants  $f_{\underline{d}}^{bc} \simeq f_{\underline{\beta}}^{\alpha\beta}$  (in our further identifications with spacetime objects like frames and connections we shall use Greek indices). The result of calculation of variations of (1.54), by using  $g_a$ , is

$$\begin{aligned} \delta q_{\mu, \underline{a}}^1 &= \frac{\partial \gamma_{\underline{a}}^1}{\partial u^\mu} - f_{\underline{a}}^{bc} \gamma_{\underline{b}}^1 q_{\mu, \underline{c}}^1, \quad \delta Q_\tau = \theta^{\mu\nu} \partial_\mu \gamma_{\underline{a}}^1 \partial_\nu q_{\tau, \underline{b}}^1 I^{\underline{a}} I^{\underline{b}} + \dots, \\ \delta q_{\mu, \underline{ab}}^2 &= \partial_\mu \gamma_{\underline{ab}}^2 - \theta^{\nu\tau} \partial_\nu \gamma_{\underline{a}}^1 \partial_\tau q_{\mu, \underline{b}}^1 - 2f_{\underline{a}}^{bc} \{ \gamma_{\underline{b}}^1 q_{\mu, \underline{cd}}^2 + \gamma_{\underline{bd}}^2 q_{\mu, \underline{c}}^1 \}. \end{aligned}$$

Let us introduce the objects  $\varepsilon$ , taking the values in de Sitter Lie algebra and  $W_\mu$ , taking values in the enveloping de Sitter algebra, i. e.  $\varepsilon = \gamma_{\underline{a}}^1 I^{\underline{a}}$  and  $W_\mu = q_{\mu, \underline{ab}}^2 I^{\underline{a}} I^{\underline{b}}$ , with the variation  $\delta W_\mu$  satisfying the equation

$$\delta W_\mu = \partial_\mu (\gamma_{\underline{ab}}^2 I^{\underline{a}} I^{\underline{b}}) - \frac{1}{2} \theta^{\tau\lambda} \{ \partial_\tau \varepsilon, \partial_\lambda q_\mu \} + i[\varepsilon, W_\mu] + i[(\gamma_{\underline{ab}}^2 I^{\underline{a}} I^{\underline{b}}), q_\nu].$$

This equation can be solved in the form

$$\gamma_{\underline{ab}}^2 = \frac{1}{2} \theta^{\nu\mu} (\partial_\nu \gamma_{\underline{a}}^1) q_{\mu, \underline{b}}^1, \quad q_{\mu, \underline{ab}}^2 = -\frac{1}{2} \theta^{\nu\tau} q_{\nu, \underline{a}}^1 \left( \partial_\tau q_{\mu, \underline{b}}^1 + R_{\tau\mu, \underline{b}}^1 \right).$$

The values  $R_{\tau\mu, \underline{b}}^1 = \partial_\tau q_{\mu, \underline{b}}^1 - \partial_\mu q_{\tau, \underline{b}}^1 + f_{\underline{a}}^{ec} q_{\tau, \underline{e}}^1 q_{\mu, \underline{c}}^1$  could be identified with the coefficients  $\mathcal{R}_{\beta\mu\nu}^\alpha$  of de Sitter nonlinear gauge gravity curvature (see



formula (1.50)) if in the commutative limit  $q_{\mu,\underline{b}}^1 \simeq \begin{pmatrix} \Gamma_{\underline{b}}^{\underline{a}} & l_0^{-1}\chi_{\underline{a}}^{\underline{a}} \\ l_0^{-1}\chi_{\underline{b}}^{\underline{a}} & 0 \end{pmatrix}$  (see (1.48)).

We note that the below presented procedure can be generalized to all the higher powers of  $\theta$ . As an example, we compute the first order corrections to the gravitational curvature:

### Noncommutative covariant gauge gravity dynamics

The constructions from the previous subsection can be summarized by a conclusion that the de Sitter algebra valued object  $\varepsilon = \gamma_{\underline{a}}^1(u) I^{\underline{a}}$  determines all the terms in the enveloping algebra  $\gamma = \gamma_{\underline{a}}^1 I^{\underline{a}} + \frac{1}{4}\theta^{\nu\mu}\partial_\nu\gamma_{\underline{a}}^1 q_{\mu,\underline{b}}^1 (I^{\underline{a}}I^{\underline{b}} + I^{\underline{b}}I^{\underline{a}}) + \dots$  and the gauge transformations are defined by  $\gamma_{\underline{a}}^1(u)$  and  $q_{\mu,\underline{b}}^1(u)$ , when  $\delta_{\gamma^1}\psi = i\gamma(\gamma^1, q_\mu^1) * \psi$ . We compute

$$\begin{aligned} [\gamma, * \zeta] &= i\gamma_{\underline{a}}^1 \zeta_{\underline{b}}^1 f_{\underline{c}}^{ab} I^{\underline{c}} + \frac{i}{2}\theta^{\nu\mu}\{\partial_\nu(\gamma_{\underline{a}}^1 \zeta_{\underline{b}}^1 f_{\underline{c}}^{ab})\} q_{\mu,\underline{c}} \\ &\quad + \left(\gamma_{\underline{a}}^1 \partial_\nu \zeta_{\underline{b}}^1 - \zeta_{\underline{a}}^1 \partial_\nu \gamma_{\underline{b}}^1\right) q_{\mu,\underline{b}} f_{\underline{c}}^{ab} + 2\partial_\nu \gamma_{\underline{a}}^1 \partial_\mu \zeta_{\underline{b}}^1 \{I^{\underline{a}}I^{\underline{b}}\}, \end{aligned}$$

where we used the properties that, for the de Sitter enveloping algebras, one holds the general formula for compositions of two transformations  $\delta_\gamma\delta_\zeta - \delta_\zeta\delta_\gamma = \delta_{i(\zeta*\gamma - \gamma*\zeta)}$ . This is also true for the restricted transformations defined by  $\gamma^1, \delta_{\gamma^1}\delta_{\zeta^1} - \delta_{\zeta^1}\delta_{\gamma^1} = \delta_{i(\zeta^1*\gamma^1 - \gamma^1*\zeta^1)}$ .

Such commutators could be used for definition of tensors

$$\hat{S}^{\mu\nu} = [\hat{U}^\mu, \hat{U}^\nu] - i\hat{\theta}^{\mu\nu}, \quad (1.55)$$

where  $\hat{\theta}^{\mu\nu}$  is respectively stated for the canonical, Lie and quantum plane structures. Under the general enveloping algebra one holds the transform  $\delta\hat{S}^{\mu\nu} = i[\hat{\gamma}, \hat{S}^{\mu\nu}]$ . For instance, the canonical case is characterized by

$$\begin{aligned} S^{\mu\nu} &= i\theta^{\mu\tau}\partial_\tau\Gamma^\nu - i\theta^{\nu\tau}\partial_\tau\Gamma^\mu + \Gamma^\mu * \Gamma^\nu - \Gamma^\nu * \Gamma^\mu \\ &= \theta^{\mu\tau}\theta^{\nu\lambda}\{\partial_\tau Q_\lambda - \partial_\lambda Q_\tau + Q_\tau * Q_\lambda - Q_\lambda * Q_\tau\}. \end{aligned}$$

We introduce the gravitational gauge strength (curvature)

$$R_{\tau\lambda} = \partial_\tau Q_\lambda - \partial_\lambda Q_\tau + Q_\tau * Q_\lambda - Q_\lambda * Q_\tau, \quad (1.56)$$

which could be treated as a noncommutative extension of de Sitter nonlinear gauge gravitational curvature (1.50), and calculate

$$R_{\tau\lambda,\underline{a}} = R_{\tau\lambda,\underline{a}}^1 + \theta^{\mu\nu}\{R_{\tau\mu,\underline{a}}^1 R_{\lambda\nu,\underline{b}}^1 - \frac{1}{2}q_{\mu,\underline{a}}^1 [(D_\nu R_{\tau\lambda,\underline{b}}^1) + \partial_\nu R_{\tau\lambda,\underline{b}}^1]\} I^{\underline{b}},$$

where the gauge gravitation covariant derivative is introduced,

$$(D_\nu R_{\tau\lambda,\underline{b}}^1) = \partial_\nu R_{\tau\lambda,\underline{b}}^1 + q_{\nu,\underline{c}} R_{\tau\lambda,\underline{d}}^1 f_{\underline{b}}^{cd}.$$

Following the gauge transformation laws for  $\gamma$  and  $q^1$  we find  $\delta_{\gamma^1} R_{\tau\lambda}^1 = i [\gamma, * R_{\tau\lambda}^1]$  with the restricted form of  $\gamma$ .

One can be formulated a gauge covariant gravitational dynamics of non-commutative spaces following the nonlinear realization of de Sitter algebra and the  $*$ -formalism and introducing derivatives in such a way that one does not obtain new relations for the coordinates. In this case, a Leibniz rule can be defined that  $\widehat{\partial}_\mu \widehat{u}^\nu = \delta_\mu^\nu + d_{\mu\sigma}^{\nu\tau} \widehat{u}^\sigma \widehat{\partial}_\tau$ , where the coefficients  $d_{\mu\sigma}^{\nu\tau} = \delta_\sigma^\nu \delta_\mu^\tau$  are chosen to have not new relations when  $\widehat{\partial}_\mu$  acts again to the right hand side. One holds the  $*$ -derivative formulas

$$\partial_\tau * f = \frac{\partial}{\partial u^\tau} f + f * \partial_\tau, \quad [\partial_l, *(f * g)] = ([\partial_l, *f]) * g + f * ([\partial_l, *g])$$

and the Stokes theorem  $\int [\partial_l, f] = \int d^N u [\partial_l, *f] = \int d^N u \frac{\partial}{\partial u^\tau} f = 0$ , where, for the canonical structure, the integral is defined,  $\int \widehat{f} = \int d^N u f(u^1, \dots, u^N)$ .

An action can be introduced by using such integrals. For instance, for a tensor of type (1.55), when  $\delta \widehat{L} = i [\widehat{\gamma}, \widehat{L}]$ , we can define a gauge invariant action  $W = \int d^N u \text{Tr} \widehat{L}$ ,  $\delta W = 0$ , where the trace has to be taken for the group generators. For the nonlinear de Sitter gauge gravity a proper action is  $L = \frac{1}{4} R_{\tau\lambda} R^{\tau\lambda}$ , where  $R_{\tau\lambda}$  is defined by (1.56) (in the commutative limit we shall obtain the connection (1.48)). In this case the dynamic of noncommutative space is entirely formulated in the framework of quantum field theory of gauge fields. In general, we are dealing with anisotropic gauge gravitational interactions. The method works for matter fields as well to restrictions to the general relativity theory.

## Noncommutative symmetries and star product deformations

The aim of this subsection is to prove that there are possible extensions of exact solutions from the Einstein and gauge gravity possessing hidden noncommutative symmetries without introducing new fields. For simplicity, we present the formulas including decompositions up to the second order on noncommutative parameter  $\theta^{\alpha\beta}$  for vielbeins, connections and curvatures which can be arranged to result in different models of noncommutative gravity. We give the data for the  $SU(1, n+m-1)$  and  $SO(1, n+m-1)$  gauge models containing, in general, complex N-elongated frames, modelling some exact solutions. All data can be considered for extensions with nonlinear

realizations into a bundle of affine/or de Sitter frames (in this case, one generates noncommutative gauge theories of type [1]) or to impose certain constraints and breaking symmetries.

The standard approaches to noncommutative geometry also contain certain noncommutative relations for coordinates,

$$[u^\alpha, u^\beta] = u^\alpha u^\beta - u^\beta u^\alpha = i\theta^{\alpha\beta}(u^\gamma) \quad (1.57)$$

were, in the simplest models, the commutator  $[u^\alpha, u^\beta]$  is approximated to be constant, but there were elaborated approaches for general manifolds with the noncommutative parameter  $\theta^{\alpha\beta}$  treated as functions on  $u^\gamma$ . We define the star (Moyal) product to include possible N-elongated partial derivatives and a quantum constant  $\hbar$ ,

$$\begin{aligned} f * \varphi &= f\varphi + \frac{\hbar}{2} B^{\bar{\alpha}\bar{\beta}} \left( \delta_{\bar{\alpha}} f \delta_{\bar{\beta}} \varphi + \delta_{\bar{\beta}} f \delta_{\bar{\alpha}} \varphi \right) + \hbar^2 B^{\bar{\alpha}\bar{\beta}} B^{\bar{\gamma}\bar{\mu}} [\delta_{(\bar{\alpha}} \delta_{\bar{\gamma})} f] [\delta_{(\bar{\beta}} \delta_{\bar{\mu})} \varphi] \\ &\quad + \frac{2}{3} \hbar^2 B^{\bar{\alpha}\bar{\beta}} \delta_{\bar{\beta}} B^{\bar{\gamma}\bar{\mu}} \{ [\delta_{(\bar{\alpha}} \delta_{\bar{\gamma})} f] \delta_{\bar{\mu}} \varphi + [\delta_{(\bar{\alpha}} \delta_{\bar{\gamma})} \varphi] \delta_{\bar{\mu}} f \} + O(\hbar^3), \end{aligned} \quad (1.58)$$

where, for instance,  $\delta_{(\mu} \delta_{\nu)} = (1/2)(\delta_\mu \delta_\nu + \delta_\nu \delta_\mu)$ ,

$$B^{\bar{\alpha}\bar{\beta}} = \frac{\theta^{\alpha\beta}}{2} \left( \delta_\alpha u^{\bar{\alpha}} \delta_\beta u^{\bar{\beta}} + \delta_\beta u^{\bar{\alpha}} \delta_\alpha u^{\bar{\beta}} \right) + O(\hbar^3) \quad (1.59)$$

is defined for new coordinates  $u^{\bar{\alpha}} = u^{\bar{\alpha}}(u^\alpha)$  inducing a suitable Poisson bi-vector field  $B^{\bar{\alpha}\bar{\beta}}(\hbar)$  being related to a quantum diagram formalism (we shall not consider details concerning geometric quantization in this paper by investigating only classicassical deformations related to any anholonomic frame and coordinate (1.57) noncommutativity origin). The formulas (1.58) and (1.59) transform into the usual ones with partial derivatives  $\partial_\alpha$  and  $\partial_{\bar{\alpha}}$  for vanishing anholonomy coefficients. We can define a star product being invariant under diffeomorphism transforms,  $* \rightarrow *^{[-]}$ , adapted to the N-connection structure (in a vector bundle provided with N-connection configuration, we use the label  $[-]$  in order to emphasize the dependence on coordinates  $u^{\bar{\alpha}}$  with 'overlined' indices), by introducing the transforms

$$f^{[-]}(\hbar) = \Theta f(\hbar), \quad f^{[-]} *^{[-]} \varphi^{[-]} = \Theta \left( \Theta^{-1} f^{[-]} * \Theta^{-1} \right) \varphi^{[-]},$$

for  $\Theta = 1 + \sum_{[k=1]} \hbar^k \Theta_{[k]}$ , for simplicity, computed up to the squared order on  $\hbar$ ,  $\Theta = 1 - 2\hbar^2 \theta^{\mu\nu} \theta^{\rho\sigma} [\delta_{(\mu} \delta_{\nu)} u^{\bar{\alpha}}] [\delta_{(\rho} \delta_{\sigma)} u^{\bar{\beta}}] \delta_{(\bar{\alpha}} \delta_{\bar{\beta})} + [\delta_{(\mu} \delta_{\rho)} u^{\bar{\alpha}}] (\delta_{\nu} u^{\bar{\beta}}) (\delta_{\sigma} u^{\bar{\gamma}})$

$\left[\delta_{(\overline{\alpha}\overline{\beta}\overline{\gamma})}\right] + O(\hbar^4)$ , where  $\delta_{(\overline{\alpha}\overline{\beta}\overline{\gamma})} = (1/3!)(\delta_{\overline{\alpha}}\delta_{\overline{\beta}}\delta_{\overline{\gamma}} + \text{all symmetric permutations})$ . In our further constructions we shall omit the constant  $\hbar$  considering that  $\theta \sim \hbar$  is a small value by writing the necessary terms in the approximation  $O(\theta^3)$  or  $O(\theta^4)$ .

We consider a noncommutative gauge theory on a space with N-connection structure stated by the gauge fields  $\hat{A}_\mu = (\hat{A}_i, \hat{A}_a)$  when "hats" on symbols will be used for the objects defined on spaces with coordinate non-commutativity. In general, the gauge model can be with different types of structure groups like  $SL(k, \mathbb{C})$ ,  $SU_k$ ,  $U_k$ ,  $SO(k-1, 1)$  and their nonlinear realizations. For instance, for the  $U(n+m)$  gauge fields there are satisfied the conditions  $\hat{A}_\mu^+ = -\hat{A}_\mu$ , where "+" is the Hermitian conjugation. It is useful to present the basic geometric constructions for a unitary structural group containing the  $SO(4, 1)$  as a particular case if we want to consider noncommutative extensions of 4D exact solutions.

The noncommutative gauge transforms of potentials are defined by using the star product  $\hat{A}_\mu^{[\varphi]} = \hat{\varphi} * \hat{A}_\mu \hat{\varphi}_{[*]}^{-1} - \hat{\varphi} * \delta_\mu \hat{\varphi}_{[*]}^{-1}$ , where the N-elongated partial derivatives are used and  $\hat{\varphi} * \hat{\varphi}_{[*]}^{-1} = 1 = \hat{\varphi}_{[*]}^{-1} * \hat{\varphi}$ . The matrix coefficients of fields will be distinguished by "overlined" indices, for instance,  $\hat{A}_\mu = \{\hat{A}_\mu^{\underline{\alpha}\underline{\beta}}\}$ , and for commutative values,  $A_\mu = \{A_\mu^{\underline{\alpha}\underline{\beta}}\}$ . Such fields are subjected to the conditions  $(\hat{A}_\mu^{\underline{\alpha}\underline{\beta}})^+(u, \theta) = -\hat{A}_\mu^{\underline{\beta}\underline{\alpha}}(u, \theta)$  and  $\hat{A}_\mu^{\underline{\alpha}\underline{\beta}}(u, -\theta) = -\hat{A}_\mu^{\underline{\beta}\underline{\alpha}}(u, \theta)$ . There is a basic assumption that the noncommutative fields are related to the commutative fields by the Seiberg-Witten map in a manner that there are not new degrees of freedom being satisfied the equation

$$\hat{A}_\mu^{\underline{\alpha}\underline{\beta}}(A) + \Delta_{\hat{\lambda}} \hat{A}_\mu^{\underline{\alpha}\underline{\beta}}(A) = \hat{A}_\mu^{\underline{\alpha}\underline{\beta}}(A + \Delta_{\hat{\lambda}} A), \quad (1.60)$$

where  $\hat{A}_\mu^{\underline{\alpha}\underline{\beta}}(A)$  denotes a functional dependence on commutative field  $A_\mu^{\underline{\alpha}\underline{\beta}}$ ,  $\hat{\varphi} = \exp \hat{\lambda}$  and the infinitesimal deformations  $\hat{A}_\mu^{\underline{\alpha}\underline{\beta}}(A)$  and of  $A_\mu^{\underline{\alpha}\underline{\beta}}$  are

$$\begin{aligned} \Delta_{\hat{\lambda}} \hat{A}_\mu^{\underline{\alpha}\underline{\beta}} &= \delta_\mu \hat{\lambda}^{\underline{\alpha}\underline{\beta}} + \hat{A}_\mu^{\underline{\alpha}\underline{\gamma}} * \hat{\lambda}^{\underline{\gamma}\underline{\beta}} - \hat{\lambda}^{\underline{\alpha}\underline{\gamma}} * \hat{A}_\mu^{\underline{\gamma}\underline{\beta}} \\ \text{and } \Delta_{\lambda} A_\mu^{\underline{\alpha}\underline{\beta}} &= \delta_\mu \lambda^{\underline{\alpha}\underline{\beta}} + A_\mu^{\underline{\alpha}\underline{\gamma}} * \lambda^{\underline{\gamma}\underline{\beta}} - \lambda^{\underline{\alpha}\underline{\gamma}} * A_\mu^{\underline{\gamma}\underline{\beta}}, \end{aligned}$$

where instead of partial derivatives  $\partial_\mu$  we use the N-elongated ones,  $\delta_\mu$  and sum on index  $\underline{\gamma}$ .

Solutions of the Seiberg-Witten equations for models of gauge gravity are considered, for instance, in Ref. [1] (there are discussed procedures of deriving expressions on  $\theta$  to all orders). Here we present only the first order

on  $\theta$  for the coefficients  $\widehat{\lambda}^{\underline{\alpha}\underline{\beta}}$  and the first and second orders for  $\widehat{A}_\mu^{\underline{\alpha}\underline{\beta}}$  including anholonomy relations and not depending on model considerations,

$$\widehat{\lambda}^{\underline{\alpha}\underline{\beta}} = \lambda^{\underline{\alpha}\underline{\beta}} + \frac{i}{4}\theta^{\nu\tau}\{(\delta_\nu\lambda^{\underline{\alpha}\underline{\gamma}})A_\mu^{\underline{\gamma}\underline{\beta}} + A_\mu^{\underline{\alpha}\underline{\gamma}}(\delta_\nu\lambda^{\underline{\gamma}\underline{\beta}})\} + O(\theta^2)$$

$$\begin{aligned} \text{and } \widehat{A}_\mu^{\underline{\alpha}\underline{\beta}} &= A_\mu^{\underline{\alpha}\underline{\beta}} - \frac{i}{4}\theta^{\nu\tau}\{A_\mu^{\underline{\alpha}\underline{\gamma}}(\delta_\tau A_\nu^{\underline{\gamma}\underline{\beta}} + R^{\underline{\gamma}\underline{\beta}}_{\tau\nu}) + (\delta_\tau A_\mu^{\underline{\alpha}\underline{\gamma}} + R^{\underline{\alpha}\underline{\gamma}}_{\tau\mu})A_\nu^{\underline{\gamma}\underline{\beta}}\} \\ &+ \frac{1}{32}\theta^{\nu\tau}\theta^{\rho\sigma}\{[2A_\rho^{\underline{\alpha}\underline{\gamma}}(R^{\underline{\gamma}\underline{\varepsilon}}_{\sigma\nu}R^{\underline{\varepsilon}\underline{\beta}}_{\mu\tau} + R^{\underline{\gamma}\underline{\varepsilon}}_{\mu\tau}R^{\underline{\varepsilon}\underline{\beta}}_{\sigma\nu}) \\ &+ 2(R^{\underline{\alpha}\underline{\varepsilon}}_{\sigma\nu}R^{\underline{\varepsilon}\underline{\gamma}}_{\mu\tau} + R^{\underline{\alpha}\underline{\varepsilon}}_{\mu\tau}R^{\underline{\varepsilon}\underline{\gamma}}_{\sigma\nu})A_\rho^{\underline{\gamma}\underline{\beta}}] \\ &- [A_\nu^{\underline{\alpha}\underline{\gamma}}(D_\tau R^{\underline{\gamma}\underline{\beta}}_{\sigma\mu} + \delta_\tau R^{\underline{\gamma}\underline{\beta}}_{\sigma\mu}) + (D_\tau R^{\underline{\alpha}\underline{\gamma}}_{\sigma\mu} + \delta_\tau R^{\underline{\alpha}\underline{\gamma}}_{\sigma\mu})A_\nu^{\underline{\gamma}\underline{\beta}}] \\ &- \delta_\sigma[A_\nu^{\underline{\alpha}\underline{\gamma}}(\delta_\tau A_\mu^{\underline{\gamma}\underline{\beta}} + R^{\underline{\gamma}\underline{\beta}}_{\tau\mu}) + (\delta_\tau A_\mu^{\underline{\alpha}\underline{\gamma}} + R^{\underline{\alpha}\underline{\gamma}}_{\tau\mu})A_\nu^{\underline{\gamma}\underline{\beta}}] + \\ &[(\delta_\nu A_\rho^{\underline{\alpha}\underline{\gamma}})(2\delta_{(\tau}\delta_{\sigma)}A_\mu^{\underline{\gamma}\underline{\beta}} + \delta_\tau R^{\underline{\gamma}\underline{\beta}}_{\sigma\mu}) + (2\delta_{(\tau}\delta_{\sigma)}A_\mu^{\underline{\alpha}\underline{\gamma}} + \delta_\tau R^{\underline{\alpha}\underline{\gamma}}_{\sigma\mu})(\delta_\nu A_\rho^{\underline{\gamma}\underline{\beta}})] - \\ &[A_\nu^{\underline{\alpha}\underline{\varepsilon}}(\delta_\tau A_\rho^{\underline{\varepsilon}\underline{\gamma}} + R^{\underline{\varepsilon}\underline{\gamma}}_{\tau\rho}) + (\delta_\tau A_\rho^{\underline{\alpha}\underline{\varepsilon}} + R^{\underline{\alpha}\underline{\varepsilon}}_{\tau\rho})A_\nu^{\underline{\varepsilon}\underline{\gamma}}](\delta_\sigma A_\mu^{\underline{\gamma}\underline{\beta}} + R^{\underline{\gamma}\underline{\beta}}_{\sigma\mu}) - \\ &(\delta_\sigma A_\mu^{\underline{\alpha}\underline{\gamma}} + R^{\underline{\alpha}\underline{\gamma}}_{\sigma\mu})[A_\nu^{\underline{\gamma}\underline{\varepsilon}}(\delta_\tau A_\rho^{\underline{\varepsilon}\underline{\beta}} + R^{\underline{\varepsilon}\underline{\beta}}_{\tau\rho}) + (\delta_\tau A_\rho^{\underline{\gamma}\underline{\varepsilon}} + R^{\underline{\gamma}\underline{\varepsilon}}_{\tau\rho})A_\nu^{\underline{\varepsilon}\underline{\beta}}] + O(\theta^3), \end{aligned} \quad (1.61)$$

where the curvature is defined  $R^{\underline{\alpha}\underline{\beta}}_{\tau\nu} = e^\alpha_{\underline{\alpha}}e^{\underline{\beta}\underline{\beta}}_{\underline{\beta}}R^\alpha_{\beta\tau\nu}$ , when  $\Gamma \rightarrow A$ , and for the gauge model of gravity, see (1.50) and (1.56). By using the star product, we can write symbolically the solution (1.61) in general form,

$$\Delta\widehat{A}_\mu^{\underline{\alpha}\underline{\beta}}(\theta) = -\frac{i}{4}\theta^{\nu\tau}\left[\widehat{A}_\mu^{\underline{\alpha}\underline{\gamma}} * (\delta_\tau\widehat{A}_\nu^{\underline{\gamma}\underline{\beta}} + \widehat{R}^{\underline{\gamma}\underline{\beta}}_{\tau\nu}) + (\delta_\tau\widehat{A}_\mu^{\underline{\alpha}\underline{\gamma}} + \widehat{R}^{\underline{\alpha}\underline{\gamma}}_{\tau\mu}) * \widehat{A}_\nu^{\underline{\gamma}\underline{\beta}}\right],$$

where  $\widehat{R}^{\underline{\gamma}\underline{\beta}}_{\tau\nu}$  is defined by the same formulas as  $R^{\underline{\alpha}\underline{\beta}}_{\tau\nu}$  but with the star products, like  $AA \rightarrow A * A$ .

There is a problem how to determine the dependence of the noncommutative vielbeins  $\widehat{e}^\alpha_{\underline{\alpha}}$  on commutative ones  $e^\alpha_{\underline{\alpha}}$ . If we consider the frame fields to be included into a (anti) de Sitter gauge gravity model with the connection (1.48), the vielbein components should be treated as certain coefficients of the gauge potential with specific nonlinear transforms for which the results of Ref. [1] hold. The main difference (considered in this work) is that the frames are in general with anholonomy induced by a N-connection field. In order to derive in a such model the Einstein gravity we have to analyze the reduction (1.53) to a Poincaré gauge gravity.

An explicit calculus of the curvature of such gauge potential show that the coefficients of the curvature of (1.53), obtained as a reduction from the  $SO(4,1)$  gauge group is given by the coefficients (1.50) with vanishing torsion and constraints of type  $\widehat{A}_\nu^{\underline{\gamma}\underline{5}} = \epsilon\widehat{e}_\nu^{\underline{\gamma}}$  and  $\widehat{A}_\nu^{\underline{5}\underline{5}} = \epsilon\widehat{\phi}_\nu$  with  $\widehat{R}^{\underline{5}\underline{5}}_{\tau\nu} \sim \epsilon$

vanishing in the limit  $\epsilon \rightarrow 0$  (we obtain the same formulas for the vielbein and curvature components derived for the inhomogeneous Lorentz group but generalized to N-elongated derivatives and with distinguishing into h-v-components). The result for  $\widehat{e}_\mu^\mu$  in the limit  $\epsilon \rightarrow 0$  generalized to the case of canonical connections defining the covariant derivatives  $D_\tau$  and corresponding curvatures is

$$\begin{aligned} \widehat{e}_\mu^\mu &= e_\mu^\mu - \frac{i}{4} \theta^{\nu\tau} \left[ A_\nu^{\mu\gamma} \delta_\tau e_\mu^\gamma + \left( \delta_\tau A_\mu^{\mu\gamma} + R^{\mu\gamma}_{\tau\mu} \right) e_\nu^\gamma \right] + \\ &\frac{1}{32} \theta^{\nu\tau} \theta^{\beta\sigma} \{ 2(R^{\mu\epsilon}_{\sigma\nu} R^{\epsilon\gamma}_{\mu\tau} + R^{\mu\epsilon}_{\mu\tau} R^{\epsilon\gamma}_{\sigma\nu}) e_\beta^\gamma - A_\beta^{\mu\gamma} (D_\tau R^{\gamma\beta}_{\sigma\mu} + \delta_\tau R^{\gamma\beta}_{\sigma\mu}) e_\beta^\beta - \\ &- [A_\nu^{\mu\gamma} (D_\tau R^{\gamma\beta}_{\sigma\mu} + \delta_\tau R^{\gamma\beta}_{\sigma\mu}) + (D_\tau R^{\mu\gamma}_{\sigma\mu} + \delta_\tau R^{\mu\gamma}_{\sigma\mu}) A_\nu^{\gamma\beta}] e_\beta^\beta - \\ &e_\beta^\beta \delta_\sigma [A_\nu^{\mu\gamma} (\delta_\tau A_\mu^{\gamma\beta} + R^{\gamma\beta}_{\tau\mu}) + (\delta_\tau A_\mu^{\mu\gamma} + R^{\mu\gamma}_{\tau\mu}) A_\nu^{\gamma\beta}] + 2(\delta_\nu A_\beta^{\mu\gamma}) \delta_{(\tau} \delta_{\sigma)} e_\mu^\gamma - \\ &- A_\beta^{\mu\gamma} \delta_\sigma [A_\nu^{\gamma\beta} \delta_\tau e_\mu^\beta + (\delta_\tau A_\mu^{\gamma\beta} + R^{\gamma\beta}_{\tau\mu}) e_\nu^\beta] - (\delta_\nu e_\beta^\gamma) \delta_\tau (\delta_\sigma A_\mu^{\mu\gamma} + R^{\mu\gamma}_{\sigma\mu}) - \\ &[A_\nu^{\mu\gamma} (\delta_\tau A_\beta^{\gamma\beta} + R^{\gamma\beta}_{\tau\beta}) + (\delta_\tau A_\beta^{\mu\gamma} + R^{\mu\gamma}_{\tau\beta}) A_\nu^{\gamma\beta}] \delta_\sigma e_\mu^\beta - \\ &(\delta_\sigma A_\mu^{\mu\gamma} + R^{\mu\gamma}_{\sigma\mu}) [A_\mu^{\gamma\beta} (\delta_\nu e_\beta^\beta) + e_\nu^\beta (\delta_\sigma A_\mu^{\gamma\beta} + R^{\gamma\beta}_{\sigma\mu})] \} + O(\theta^3). \end{aligned} \quad (1.62)$$

Having the decompositions (1.62), we can define the inverse vielbein  $\widehat{e}_{*\underline{\mu}}^\mu$  from the equation  $\widehat{e}_{*\underline{\mu}}^\mu * \widehat{e}_\mu^\nu = \delta_{\underline{\mu}}^\nu$  and consequently compute  $\theta$ -deformations of connections, curvatures, torsions and any type of actions and field equations (for simplicity, we omit such cumbersome formulas).

### 1.2.6 Exact solutions for (non)commutative Finsler branes

We show how the anholonomic deformation method can be applied for generating Finsler like solutions, with nontrivial nonlinear connection structure, in noncommutative gravity [8].

#### Nonholonomic Distributions and Noncommutative Gravity

There exist many formulations of noncommutative geometry/gravity based on nonlocal deformation of spacetime and field theories starting from noncommutative relations of type

$$u^\alpha u^\beta - u^\beta u^\alpha = i\theta^{\alpha\beta}, \quad (1.63)$$

where  $u^\alpha$  are local spacetime coordinates,  $i$  is the imaginary unity,  $i^2 = -1$ , and  $\theta^{\alpha\beta}$  is an anti-symmetric second-rank tensor (which, for simplicity, for certain models, is taken to be with constant coefficients). Following

our unified approach to (pseudo) Riemannian and Finsler–Lagrange spaces (using the geometry of nonholonomic manifolds) we consider that for  $\theta^{\alpha\beta} \rightarrow 0$  the local coordinates  $u^\alpha$  are on a four dimensional (4-d) nonholonomic manifold  $\mathbf{V}$  of necessary smooth class. Such spacetimes can be enabled with a conventional 2+2 splitting (defined by a nonholonomic, equivalently, anholonomic/ non-integrable real distribution), when local coordinates  $u = (x, y)$  on an open region  $U \subset \mathbf{V}$  are labelled in the form  $u^\alpha = (x^i, y^a)$ , with indices of type  $i, j, k, \dots = 1, 2$  and  $a, b, c, \dots = 3, 4$ . The coefficients of tensor like objects on  $\mathbf{V}$  can be computed with respect to a general (non-coordinate) local basis  $e_\alpha = (e_i, e_a)$ .<sup>29</sup>

On a commutative  $\mathbf{V}$ , any (prime) metric  $\mathbf{g} = \mathbf{g}_{\alpha\beta} \mathbf{e}^\alpha \otimes \mathbf{e}^\beta$  (for instance, a Schwarzschild, ellipsoid, ring or other type solution, their conformal transforms and nonholonomic deformations which, in general, are not solutions of the Einstein equations) can be parametrized in the form

$$\begin{aligned} \mathbf{g} &= g_i(u) dx^i \otimes dx^i + h_a(u) \mathbf{e}^a \otimes \mathbf{e}^a, \\ \mathbf{e}^\alpha &= \mathbf{e}^\alpha_{\underline{\alpha}}(u) du^{\underline{\alpha}} = (e^i = dx^i, \mathbf{e}^a = dy^a + N_i^a dx^i). \end{aligned} \quad (1.64)$$

It is convenient to work with the so-called canonical distinguished connection (in brief, canonical d-connection  $\hat{\mathbf{D}} = \{\hat{\Gamma}^\gamma_{\alpha\beta}\}$ ) which is metric compatible,  $\hat{\mathbf{D}}\mathbf{g} = 0$ , and completely defined by the coefficients of a metric  $\mathbf{g}$  (1.64) and a N-connection  $\mathbf{N}$ , subjected to the condition that the so-called  $h$ - and  $v$ -components of torsion are zero.<sup>30</sup> Using formula  $\Gamma^\gamma_{\alpha\beta} = \hat{\Gamma}^\gamma_{\alpha\beta} + Z^\gamma_{\alpha\beta}$ , where  $\nabla = \{\Gamma^\gamma_{\alpha\beta}\}$  is the Levi-Civita connection (this connection is metric compatible, torsionless and completely defined by the coefficients of the same metric structure  $\mathbf{g}$ ), we can perform all geometric constructions in two equivalent forms: applying the covariant derivative  $\hat{\mathbf{D}}$  and/or  $\nabla$ . This is possible because all values  $\Gamma$ ,  $\hat{\Gamma}$  and  $Z$  are completely determined in unique forms by  $\mathbf{g}$  for a prescribed nonholonomic splitting.

There were considered different constructions of  ${}^\theta\mathcal{A}$  corresponding to different choices of the so-called "symbols of operators" and the extended

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<sup>29</sup>If  $\mathbf{V} = TM$  is the total space of a tangent bundle  $(TM, \pi, M)$  on a two dimensional (2-d) base manifold  $M$ , the values  $x^i$  and  $y^a$  are respectively the base coordinates (on a low-dimensional space/ spacetime) and fiber coordinates (velocity like). Alternatively, we can consider that  $\mathbf{V} = V$  is a 4-d nonholonomic manifold (in particular, a pseudo-Riemannian one) with local fibered structure.

<sup>30</sup>by definition, a d-connection is a linear connection preserving under parallelism a given N-connection splitting; in general, a d-connection has a nontrivial torsion tensor but for the canonical d-connection the torsion is induced by the anholonomy coefficients which in their turn are defined by certain off-diagonal N-coefficients in the corresponding metric

Weyl ordered symbol  $\mathcal{W}$ , to get an algebra isomorphism with properties  $\mathcal{W}[^1f \star ^2f] \equiv \mathcal{W}[^1f]\mathcal{W}[^2f] = ^1\hat{f} \ ^2\hat{f}$ , for  $^1f, ^2f \in \mathcal{C}(\mathbf{V})$  and  $^1\hat{f}, ^2\hat{f} \in {}^\theta\mathcal{A}(\mathbf{V})$ , when the induced  $\star$ -product is associative and noncommutative. Such a product can be introduced on nonholonomic manifolds using the N-elongated partial derivatives,

$$^1\hat{f} \star ^2\hat{f} = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{i}{2}\right)^k \theta^{\alpha_1\beta_1} \dots \theta^{\alpha_k\beta_k} \mathbf{e}_{\alpha_1} \dots \mathbf{e}_{\alpha_k} ^1f(u) \mathbf{e}_{\beta_1} \dots \mathbf{e}_{\beta_k} ^2f(u). \quad (1.65)$$

For a noncommutative nonholonomic spacetime model  ${}^\theta\mathbf{V}$  of a spacetime  $\mathbf{V}$ , we can derive a N-adapted local frame structure  ${}^\theta\mathbf{e}_\alpha = ({}^\theta\mathbf{e}_i, {}^\theta\mathbf{e}_a)$  which can be constructed by noncommutative deformations of  $\mathbf{e}_\alpha$ ,

$$\begin{aligned} {}^\theta\mathbf{e}_\alpha^{\underline{\alpha}} &= \mathbf{e}_\alpha^{\underline{\alpha}} + i\theta^{\alpha_1\beta_1} \mathbf{e}_{\alpha\alpha_1\beta_1}^{\underline{\alpha}} + \theta^{\alpha_1\beta_1}\theta^{\alpha_2\beta_2} \mathbf{e}_{\alpha\alpha_1\beta_1\alpha_2\beta_2}^{\underline{\alpha}} + \mathcal{O}(\theta^3), \\ {}^\theta\mathbf{e}_{\star\alpha}^{\underline{\alpha}} &= \mathbf{e}_{\star\alpha}^{\underline{\alpha}} + i\theta^{\alpha_1\beta_1} \mathbf{e}_{\star\alpha\alpha_1\beta_1}^{\underline{\alpha}} + \theta^{\alpha_1\beta_1}\theta^{\alpha_2\beta_2} \mathbf{e}_{\star\alpha\alpha_1\beta_1\alpha_2\beta_2}^{\underline{\alpha}} + \mathcal{O}(\theta^3), \end{aligned} \quad (1.66)$$

subjected to the condition  ${}^\theta\mathbf{e}_{\star\alpha}^{\underline{\alpha}} \star {}^\theta\mathbf{e}_\alpha^{\underline{\beta}} = \delta_{\underline{\alpha}}^{\underline{\beta}}$ , for  $\delta_{\underline{\alpha}}^{\underline{\beta}}$  being the Kronecker tensor, where  $\mathbf{e}_{\alpha\alpha_1\beta_1}^{\underline{\alpha}}$  and  $\mathbf{e}_{\alpha\alpha_1\beta_1\alpha_2\beta_2}^{\underline{\alpha}}$  can be written in terms of  $\mathbf{e}_\alpha^{\underline{\alpha}}, \theta^{\alpha\beta}$  and the spin distinguished connection corresponding to  $\hat{\mathbf{D}}$ .

The noncommutative deformation of a metric (1.64),  $\mathbf{g} \rightarrow {}^\theta\mathbf{g}$ , can be defined in the form

$${}^\theta\mathbf{g}_{\alpha\beta} = \frac{1}{2}\eta_{\underline{\alpha}\underline{\beta}} \left[ {}^\theta\mathbf{e}_\alpha^{\underline{\alpha}} \star \left( {}^\theta\mathbf{e}_\beta^{\underline{\beta}} \right)^+ + {}^\theta\mathbf{e}_\beta^{\underline{\beta}} \star \left( {}^\theta\mathbf{e}_\alpha^{\underline{\alpha}} \right)^+ \right], \quad (1.67)$$

where  $(\dots)^+$  denotes Hermitian conjugation and  $\eta_{\underline{\alpha}\underline{\beta}}$  is the flat Minkowski space metric. In N-adapted form, as nonholonomic deformations, such metrics were used for constructing exact solutions in string/gauge/Einstein and Lagrange-Finsler metric-affine and noncommutative gravity theories.

The target metrics resulting after noncommutative nonholonomic transforms (to be investigated in this work) can be parametrized in general form

$$\begin{aligned} {}^\theta\mathbf{g} &= {}^\theta g_i(u, \theta) dx^i \otimes dx^i + {}^\theta h_a(u, \theta) {}^\theta\mathbf{e}^a \otimes {}^\theta\mathbf{e}^a, \\ {}^\theta\mathbf{e}^\alpha &= {}^\theta\mathbf{e}_{\underline{\alpha}}^\alpha(u, \theta) du^{\underline{\alpha}} = \left( e^i = dx^i, {}^\theta\mathbf{e}^a = dy^a + {}^\theta N_i^a(u, \theta) dx^i \right), \end{aligned} \quad (1.68)$$

where it is convenient to consider conventional polarizations  $\eta_{\dots}$  when

$${}^\theta g_i = \check{\eta}_i(u, \theta) g_i, \quad {}^\theta h_a = \check{\eta}_a(u, \theta) h_a, \quad {}^\theta N_i^a(u, \theta) = \check{\eta}_i^a(u, \theta) N_i^a, \quad (1.69)$$

for  $g_i, h_a, N_i^a$  given by a prime metric (1.64).



In this work, we shall analyze noncommutative deformations induced by (1.63) for a class of four dimensional, 4-d, (pseudo) Riemannian metrics (or 2-d (pseudo) Finsler metrics) defining (non) commutative Finsler–Einstein spaces as exact solutions of the Einstein equations,

$${}^\theta \widehat{E}^i_j = {}^\theta_h \Upsilon(u) \delta^i_j, \quad \widehat{E}^a_b = {}^\theta_v \Upsilon(u) \delta^a_b, \quad {}^\theta \widehat{E}_{ia} = {}^\theta \widehat{E}_{ai} = 0, \quad (1.70)$$

where  ${}^\theta \widehat{\mathbf{E}}_{\alpha\beta} = \{{}^\theta \widehat{E}_{ij}, {}^\theta \widehat{E}_{ia}, {}^\theta \widehat{E}_{ai}, {}^\theta \widehat{E}_{ab}\}$  are the components of the Einstein tensor computed for the canonical distinguished connection (d-connection)  ${}^\theta \widehat{\mathbf{D}}$ . Functions  ${}^\theta_h \Upsilon$  and  ${}^\theta_v \Upsilon$  are considered to be defined by certain matter fields in a corresponding model of (non) commutative gravity. The geometric objects in (1.70) must be computed using the  $\star$ -product (1.65) and the coefficients may contain the complex unity  $i$ . Nevertheless, it is possible to prescribe such nonholonomic distributions on the "prime"  $\mathbf{V}$  when, for instance,  $\widehat{E}^i_j(u) \rightarrow \widehat{E}^i_j(u, \theta)$ ,  ${}^\theta_h \Upsilon(u) \rightarrow {}^\theta_h \Upsilon(u, \theta), \dots$  and we get Lagrange–Finsler and/or (pseudo) Riemannian geometries, and corresponding gravitational models, with parametric dependencies of geometric objects on  $\theta$ .

Solutions of nonholonomic equations (1.70) are typical ones for the Finsler gravity with metric compatible d-connections<sup>31</sup> or in the so-called Einsteing/string/brane/gauge gravity with nonholonomic/Finsler like variables. In the standard approach to the Einstein gravity, when  $\widehat{\mathbf{D}} \rightarrow \nabla$ , the Einstein spaces are defined by metrics  $\mathbf{g}$  as solutions of the equations

$$E_{\alpha\beta} = \Upsilon_{\alpha\beta}, \quad (1.71)$$

where  $E_{\alpha\beta}$  is the Einstein tensor for  $\nabla$  and  $\Upsilon_{\alpha\beta}$  is proportional to the energy–momentum tensor of matter in general relativity. Of course, for noncommutative gravity models in (1.71), we must consider values of type  ${}^\theta \nabla$ ,  ${}^\theta E$ ,  ${}^\theta \Upsilon$  etc. Nevertheless, for certain general classes of ansatz of

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<sup>31</sup>We emphasize that Finsler like coordinates can be considered on any (pseudo), or complex Riemannian manifold and inversely. A real Finsler metric  $\mathbf{f} = \{f_{\alpha\beta}\}$  can be parametrized in the canonical Sasaki form  $\mathbf{f} = f_{ij} dx^i \otimes dx^j + f_{ab} {}^c \mathbf{e}^a \otimes {}^c \mathbf{e}^b$ ,  ${}^c \mathbf{e}^a = dy^a + {}^c N^a_i dx^i$ , where the Finsler configuration is defined by 1) a fundamental real Finsler (generating) function  $F(u) = F(x, y) = F(x^i, y^a) > 0$  if  $y \neq 0$  and homogeneous of type  $F(x, \lambda y) = |\lambda| F(x, y)$ , for any nonzero  $\lambda \in \mathbb{R}$ , with positively definite Hessian  $f_{ab} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^a \partial y^b}$ , when  $\det |f_{ab}| \neq 0$ . The Cartan canonical N-connection structure  ${}^c \mathbf{N} = \{{}^c N^a_i\}$  is defined for an effective Lagrangian  $L = F^2$  as  ${}^c N^a_i = \frac{\partial G^a}{\partial y^{2+i}}$  with  $G^a = \frac{1}{4} f^{a\ 2+i} \left( \frac{\partial^2 L}{\partial y^{2+i} \partial x^k} y^{2+k} - \frac{\partial L}{\partial x^i} \right)$ , where  $f^{ab}$  is inverse to  $f_{ab}$  and respective contractions of horizontal (h) and vertical (v) indices,  $i, j, \dots$  and  $a, b, \dots$ , are performed following the rule: we can write, for instance, an up  $v$ -index  $a$  as  $a = 2 + i$  and contract it with a low index  $i = 1, 2$ . In brief, we shall write  $y^i$  instead of  $y^{2+i}$ , or  $y^a$ .

primary metrics  $\mathbf{g}$  on a  $\mathbf{V}$  we can reparametrize such a way the nonholonomic distributions on corresponding  ${}^\theta\mathbf{V}$  that  ${}^\theta\mathbf{g}(u) = \tilde{\mathbf{g}}(u, \theta)$  are solutions of (1.70) transformed into a system of partial differential equations (with parametric dependence of coefficients on  $\theta$ ) which after certain further restrictions on coefficients determining the nonholonomic distribution can result in generic off-diagonal solutions for general relativity.<sup>32</sup>

### General solutions with noncommutative parameters

A noncommutative deformation of coordinates of type (1.63) defined by  $\theta$  together with correspondingly stated nonholonomic distributions on  ${}^\theta\mathbf{V}$  transform prime metrics  $\mathbf{g}$  (for instance, a Schwarzschild solution on  $\mathbf{V}$ ) into respective classes of target metrics  ${}^\theta\mathbf{g} = \tilde{\mathbf{g}}$  as solutions of Finsler type gravitational field equations (1.70) and/or standard Einstein equations (1.71) in general gravity. The goal of this section is to show how such solutions and their noncommutative/nonholonomic transforms can be constructed in general form for vacuum and non-vacuum locally anisotropic configurations.

We parametrize the noncommutative and nonholonomic transform of a metric  $\mathbf{g}$  (1.64) into a  ${}^\theta\mathbf{g} = \tilde{\mathbf{g}}$  (1.68) resulting from formulas (1.66), and (1.67) and expressing of polarizations in (1.69), as  $\tilde{\eta}_\alpha(u, \theta) = \dot{\eta}_\alpha(u) + \dot{\eta}_\alpha(u)\theta^2 + \mathcal{O}(\theta^4)$ , in the form

$$\begin{aligned} {}^\theta g_i &= \dot{g}_i(u) + \dot{g}_i(u)\theta^2 + \mathcal{O}(\theta^4), \quad {}^\theta h_a = \dot{h}_a(u) + \dot{h}_a(u)\theta^2 + \mathcal{O}(\theta^4), \\ {}^\theta N_i^3 &= {}^\theta w_i(u, \theta), \quad {}^\theta N_i^4 = {}^\theta n_i(u, \theta), \end{aligned} \quad (1.72)$$

where  $\dot{g}_i = g_i$  and  $\dot{h}_a = h_a$  for  $\dot{\eta}_\alpha = 1$ ; for general  $\dot{\eta}_\alpha(u)$  we get nonholonomic deformations which do not depend on  $\theta$ .

The gravitational field equations (1.70) for a metric (1.68) with coefficients (1.72) and sources of type

$${}^\theta\Upsilon_\beta^\alpha = [\Upsilon_1^1 = \Upsilon_2(x^i, v, \theta), \Upsilon_2^2 = \Upsilon_2(x^i, v, \theta), \Upsilon_3^3 = \Upsilon_4(x^i, \theta), \Upsilon_4^4 = \Upsilon_4(x^i, \theta)] \quad (1.73)$$

transform into this system of partial differential equations:

$$\begin{aligned} {}^\theta\hat{R}_1^1 &= {}^\theta\hat{R}_2^2 = \frac{1}{2 {}^\theta g_1 {}^\theta g_2} \times \\ &\left[ \frac{{}^\theta g_1^\bullet {}^\theta g_2^\bullet}{2 {}^\theta g_1} + \frac{({}^\theta g_2^\bullet)^2}{2 {}^\theta g_2} - {}^\theta g_2^{\bullet\bullet} + \frac{{}^\theta g_1' {}^\theta g_2'}{2 {}^\theta g_2} + \frac{({}^\theta g_1')^2}{2 {}^\theta g_1} - {}^\theta g_1'' \right] = -\Upsilon_4(x^i, \theta), \\ {}^\theta\hat{S}_3^3 &= {}^\theta\hat{S}_4^4 = \frac{1}{2 {}^\theta h_3 {}^\theta h_4} [{}^\theta h_4^* (\ln \sqrt{{}^\theta h_3 {}^\theta h_4})^* - {}^\theta h_4^{**}] = -\Upsilon_2(x^i, v, \theta), \\ {}^\theta\hat{R}_{3i} &= -{}^\theta w_i \frac{\beta}{2 {}^\theta h_4} - \frac{\alpha_i}{2 {}^\theta h_4} = 0, \quad {}^\theta\hat{R}_{4i} = -\frac{{}^\theta h_3}{2 {}^\theta h_4} [{}^\theta n_i^{**} + \gamma {}^\theta n_i^*] = 0, \end{aligned} \quad (1.74)$$

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<sup>32</sup>the metrics for such spacetimes can not diagonalized by coordinate transforms

where, for  ${}^\theta h_{3,4}^* \neq 0$ ,

$$\alpha_i = {}^\theta h_4^* \partial_i \phi, \quad \beta = {}^\theta h_4^* \phi^*, \quad \gamma = \frac{3 {}^\theta h_4^*}{2 {}^\theta h_4} - \frac{{}^\theta h_3^*}{{}^\theta h_3}, \quad \phi = \ln |{}^\theta h_3^* / \sqrt{|{}^\theta h_3 {}^\theta h_4|}|, \quad (1.75)$$

when the necessary partial derivatives are written in the form  $a^\bullet = \partial a / \partial x^1$ ,  $a' = \partial a / \partial x^2$ ,  $a^* = \partial a / \partial v$ . In the vacuum case, we must consider  $\Upsilon_{2,4} = 0$ . Various classes of (non) holonomic Einstein, Finsler–Einstein and generalized spaces can be generated if the sources (1.73) are taken  $\Upsilon_{2,4} = \lambda$ , where  $\lambda$  is a nonzero cosmological constant.

Let us express the coefficients of a target metric (1.68), and respective polarizations (1.69), in the form

$$\begin{aligned} {}^\theta g_k &= \epsilon_k e^{\psi(x^i, \theta)}, \\ {}^\theta h_3 &= \epsilon_3 h_0^2(x^i, \theta) [f^*(x^i, v, \theta)]^2 |\varsigma(x^i, v, \theta)|, \quad {}^\theta h_4 = \epsilon_4 [f(x^i, v, \theta) - f_0(x^i, \theta)]^2, \\ {}^\theta N_k^3 &= w_k(x^i, v, \theta), \quad {}^\theta N_k^4 = n_k(x^i, v, \theta), \end{aligned} \quad (1.76)$$

with arbitrary constants  $\epsilon_\alpha = \pm 1$ , and  $h_3^* \neq 0$  and  $h_4^* \neq 0$ , when  $f^* = 0$ . By straightforward verifications, we can prove that any off-diagonal metric

$$\begin{aligned} {}^\theta \circ \mathbf{g} &= e^\psi \epsilon_i dx^i \otimes dx^i + \epsilon_3 h_0^2 [f^*]^2 |\varsigma| \delta v \otimes \delta v + \epsilon_4 [f - f_0]^2 \delta y^4 \otimes \delta y^4, \\ \delta v &= dv + w_k(x^i, v, \theta) dx^k, \quad \delta y^4 = dy^4 + n_k(x^i, v, \theta) dx^k, \end{aligned} \quad (1.77)$$

defines an exact solution of the system of partial differential equations (1.74), i.e. of the Einstein equation for the canonical d-connection (1.70) for a metric of type (1.68) with the coefficients of form (1.76), if there are satisfied the conditions<sup>33</sup>:

1. function  $\psi$  is a solution of equation  $\epsilon_1 \psi^{\bullet\bullet} + \epsilon_2 \psi'' = \Upsilon_4$ ;
2. the value  $\varsigma$  is computed following formula

$$\varsigma(x^i, v, \theta) = \varsigma_{[0]}(x^i, \theta) - \frac{\epsilon_3}{8} h_0^2(x^i, \theta) \int \Upsilon_2 f^* [f - f_0] dv$$

and taken  $\varsigma = 1$  for  $\Upsilon_2 = 0$ ;

3. for a given source  $\Upsilon_4$ , the N-connection coefficients are computed following the formulas

$$\begin{aligned} w_i(x^k, v, \theta) &= -\partial_i \varsigma / \varsigma^*, \\ n_k(x^k, v, \theta) &= {}^1 n_k(x^i, \theta) + {}^2 n_k(x^i, \theta) \int \frac{[f^*]^2 \varsigma dv}{[f - f_0]^3}, \end{aligned}$$

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<sup>33</sup>we put the left symbol "o" in order to emphasize that such a metric is a solution of gravitational field equations

and  $w_i(x^k, v, \theta)$  are arbitrary functions if  $\varsigma = 1$  for  $\Upsilon_2 = 0$ .

It should be emphasized that such solutions depend on arbitrary non-trivial functions  $f$  (with  $f^* \neq 0$ ),  $f_0, h_0, \varsigma_{[0]}, {}^1n_k$  and  ${}^2n_k$ , and sources  $\Upsilon_2$  and  $\Upsilon_4$ . Such values for the corresponding quasi-classical limits of solutions to metrics of signatures  $\epsilon_\alpha = \pm 1$  have to be defined by certain boundary conditions and physical considerations.

Ansatz of type (1.68) for coefficients (1.76) with  $h_3^* = 0$  but  $h_4^* \neq 0$  (or, inversely,  $h_3^* \neq 0$  but  $h_4^* = 0$ ) consist more special cases and request a bit different method of constructing exact solutions.

### Off-diagonal solutions for the Levi-Civita connection

The solutions for the gravitational field equations for the canonical d-connection (which can be used for various models of noncommutative Finsler gravity and generalizations) presented in the previous subsection can be constrained additionally and transformed into solutions of the Einstein equations for the Levi-Civita connection (1.71), all depending, in general, on parameter  $\theta$ . Such classes of metrics are of type

$$\begin{aligned} \theta \circ \mathbf{g} &= e^{\psi(x^i, \theta)} [\epsilon_1 dx^1 \otimes dx^1 + \epsilon_2 dx^2 \otimes dx^2] \\ &\quad + h_3(x^i, v, \theta) \delta v \otimes \delta v + h_4(x^i, v, \theta) \delta y^4 \otimes \delta y^4, \\ \delta v &= dv + w_1(x^i, v, \theta) dx^1 + w_2(x^i, v, \theta) dx^2, \\ \delta y^4 &= dy^4 + n_1(x^i, \theta) dx^1 + n_2(x^i, \theta) dx^2, \end{aligned} \quad (1.78)$$

with the coefficients restricted to satisfy the conditions

$$\begin{aligned} \epsilon_1 \psi^{\bullet\bullet} + \epsilon_2 \psi'' &= \Upsilon_4, \quad h_4^* \phi / h_3 h_4 = \Upsilon_2, \\ w_1' - w_2^\bullet + w_2 w_1^* - w_1 w_2^* &= 0, \quad n_1' - n_2^\bullet = 0, \end{aligned}$$

for  $w_i = \partial_i \phi / \phi^*$ , see (1.75), for given sources  $\Upsilon_4(x^k, \theta)$  and  $\Upsilon_2(x^k, v, \theta)$ .

Even the ansatz (1.78) depends on three coordinates  $(x^k, v)$  and non-commutative parameter  $\theta$ , it allows us to construct more general classes of solutions with dependence on four coordinates if such metrics can be related by chains of nonholonomic transforms.

### Noncommutative deforms of the Schwarzschild metric

Solutions of type (1.77) and/or (1.78) are very general ones induced by noncommutative nonholonomic distributions and it is not clear what type of physical interpretation can be associated to such metrics. There

are analyzed certain classes of nonholonomic constraints which allows us to construct black hole solutions and noncommutative corrections.

**Vacuum noncommutative nonholonomic configurations:** In the simplest case, we analyse a class of holonomic noncommutative deformations, with  ${}^\theta N_i^a = 0$ , of the Schwarzschild metric

$$\begin{aligned} {}^{Sch} \mathbf{g} &= {}_1 g_1 dr \otimes dr + {}_1 g_2 d\vartheta \otimes d\vartheta + {}_1 h_3 d\varphi \otimes d\varphi + {}_1 h_4 dt \otimes dt, \\ {}_1 g_1 &= -\left(1 - \frac{\alpha}{r}\right)^{-1}, \quad {}_1 g_2 = -r^2, \quad {}_1 h_3 = -r^2 \sin^2 \vartheta, \quad {}_1 h_4 = 1 - \frac{\alpha}{r}, \end{aligned}$$

written in spherical coordinates  $u^\alpha = (x^1 = \xi, x^2 = \vartheta, y^3 = \varphi, y^4 = t)$  for  $\alpha = 2G\mu_0/c^2$ , correspondingly defined by the Newton constant  $G$ , a point mass  $\mu_0$  and light speed  $c$ . Taking

$$\begin{aligned} {}_1 \dot{g}_i &= {}_1 g_i, \quad {}_1 \dot{h}_a = {}_1 h_a, \quad {}_1 \dot{g}_1 = -\frac{\alpha(4r-3\alpha)}{16r^2(r-\alpha)^2}, \quad {}_1 \dot{g}_2 = -\frac{2r^2-17\alpha(r-\alpha)}{32r(r-\alpha)}, \\ {}_1 \dot{h}_3 &= -\frac{(r^2+\alpha r-\alpha^2)\cos\vartheta-\alpha(2r-\alpha)}{16r(r-\alpha)}, \quad {}_1 \dot{h}_4 = -\frac{\alpha(8r-11\alpha)}{16r^4}, \end{aligned} \quad (1.79)$$

for  ${}^\theta g_i = {}_1 \dot{g}_i + {}_1 \dot{g}_i \theta^2 + \mathcal{O}(\theta^4)$ ,  ${}^\theta h_a = {}_1 \dot{h}_a + {}_1 \dot{h}_a \theta^2 + \mathcal{O}(\theta^4)$ , we get a "degenerated" case of solutions (1.77), because  ${}^\theta h_a^* = \partial {}^\theta h_a / \partial \varphi = 0$ , which is related to the case of holonomic/ integrable off-diagonal metrics.

A more general class of noncommutative deformations of the Schwarzschild metric can be generated by nonholonomic transform of type (1.69) when the metric coefficients polarizations,  $\tilde{\eta}_\alpha$ , and N-connection coefficients,  ${}^\theta N_i^a$ , for  ${}^\theta g_i = \tilde{\eta}_i(r, \vartheta, \theta) {}_1 g_i$ ,  ${}^\theta h_a = \tilde{\eta}_a(r, \vartheta, \varphi, \theta) {}_1 h_a$ ,  ${}^\theta N_i^3 = w_i(r, \vartheta, \varphi, \theta)$ ,  ${}^\theta N_i^4 = n_i(r, \vartheta, \varphi, \theta)$ , are constrained to define a metric (1.77) for  $\Upsilon_4 = \Upsilon_2 = 0$ . The coefficients of such metrics, computed with respect to N-adapted frames defined by  ${}^\theta N_i^a$ , can be re-parametrized

$$\begin{aligned} {}^\theta g_k &= \epsilon_k e^{\psi(r, \vartheta, \theta)} = {}_1 \dot{g}_k + \delta {}_1 \dot{g}_k + ({}_1 \dot{g}_k + \delta {}_1 \dot{g}_k) \theta^2 + \mathcal{O}(\theta^4); \\ {}^\theta h_3 &= \epsilon_3 h_0^2 [f^*(r, \vartheta, \varphi, \theta)]^2 = \\ &\quad \left( {}_1 \dot{h}_3 + \delta {}_1 \dot{h}_3 \right) + \left( {}_1 \dot{h}_3 + \delta {}_1 \dot{h}_3 \right) \theta^2 + \mathcal{O}(\theta^4), \quad h_0 = \text{const} \neq 0; \\ {}^\theta h_4 &= \epsilon_4 [f(r, \vartheta, \varphi, \theta) - f_0(r, \vartheta, \theta)]^2 = ({}_1 \dot{h}_4 + \delta {}_1 \dot{h}_4) + ({}_1 \dot{h}_4 + \delta {}_1 \dot{h}_4) \theta^2 + \mathcal{O}(\theta^4), \end{aligned} \quad (1.80)$$

where the nonholonomic deformations  $\delta {}_1 \dot{g}_k, \delta {}_1 \dot{g}_k, \delta {}_1 \dot{h}_a, \delta {}_1 \dot{h}_a$  are for correspondingly given generating functions  $\psi(r, \vartheta, \theta)$  and  $f(r, \vartheta, \varphi, \theta)$  expressed as series on  $\theta^{2k}$ , for  $k = 1, 2, \dots$ . Such coefficients define noncommutative Finsler type spacetimes being solutions of the Einstein equations for the canonical d-connection. They are determined by the (prime) Schwarzschild

data  ${}_1g_i$  and  ${}_1h_a$  and certain classes on noncommutative nonholonomic distributions defining off-diagonal gravitational interactions. In order to get solutions for the Levi-Civita connection, we have to constrain (1.80) additionally in a form to generate metrics of type (1.78) with coefficients subjected to conditions (1.79) for zero sources  $\Upsilon_\alpha$ .

**Noncommutative deformations with nontrivial sources:** In the holonomic case, there are known such noncommutative generalizations of the Schwarzschild metric when

$$\begin{aligned} {}^{ncS}\mathbf{g} &= {}_\tau g_1 dr \otimes dr + {}_\tau g_2 d\vartheta \otimes d\vartheta + {}_\tau h_3 d\varphi \otimes d\varphi + {}_\tau h_4 dt \otimes dt, \\ {}_\tau g_1 &= -\left(1 - \frac{4\mu_0\gamma}{\sqrt{\pi}r}\right)^{-1}, \quad {}_\tau g_2 = -r^2, \quad {}_\tau h_3 = -r^2 \sin^2 \vartheta, \quad {}_\tau h_4 = 1 - \frac{4\mu_0\gamma}{\sqrt{\pi}r}, \end{aligned} \quad (1.81)$$

for  $\gamma$  being the so-called lower incomplete Gamma function  $\gamma(\frac{3}{2}, \frac{r^2}{4\theta}) := \int_0^{r^2} p^{1/2} e^{-p} dp$ , is the solution of noncommutative Einstein equations  ${}^\theta E_{\alpha\beta} = \frac{8\pi G}{c^2} {}^\theta T_{\alpha\beta}$ , where  ${}^\theta E_{\alpha\beta}$  is formally left unchanged (i.e. is for the commutative Levi-Civita connection in commutative coordinates) but

$${}^\theta T^\alpha_\beta = \begin{pmatrix} -p_1 & & & \\ & -p_\perp & & \\ & & -p_\perp & \\ & & & \rho_\theta \end{pmatrix}, \quad (1.82)$$

with  $p_1 = -\rho_\theta$  and  $p_\perp = -\rho_\theta - \frac{r}{2}\partial_r \rho_\theta(r)$  is taken for a self-gravitating, anisotropic fluid-type matter modeling noncommutativity.

Via nonholonomic deforms, we can generalize the solution (1.81) to off-diagonal metrics of type

$$\begin{aligned} {}^\theta {}^{ncS}\mathbf{g} &= -e^{\psi(r,\vartheta,\theta)} [dr \otimes dr + d\vartheta \otimes d\vartheta] - h_0^2 [f^*(r, \vartheta, \varphi, \theta)]^2 |\varsigma(r, \vartheta, \varphi, \theta)| \\ &\quad \delta\varphi \otimes \delta\varphi + [f(r, \vartheta, \varphi, \theta) - f_0(r, \vartheta, \theta)]^2 \delta t \otimes \delta t, \\ \delta\varphi &= d\varphi + w_1(r, \vartheta, \varphi, \theta)dr + w_2(r, \vartheta, \varphi, \theta)d\vartheta, \\ \delta t &= dt + n_1(r, \vartheta, \varphi, \theta)dr + n_2(r, \vartheta, \varphi, \theta)d\vartheta, \end{aligned} \quad (1.83)$$

being exact solutions of the Einstein equation for the canonical d-connection (1.70) with locally anisotropically self-gravitating source

$${}^\theta \Upsilon^\alpha_\beta = [\Upsilon^1_1 = \Upsilon^2_2 = \Upsilon_2(r, \vartheta, \varphi, \theta), \Upsilon^3_3 = \Upsilon^4_4 = \Upsilon_4(r, \vartheta, \theta)].$$

Such sources should be taken with certain polarization coefficients when  $\Upsilon \sim \eta T$  is constructed using the matter energy-momentum tensor (1.82).

The coefficients of metric (1.83) are computed to satisfy correspondingly the conditions:

1. function  $\psi(r, \vartheta, \theta)$  is a solution of equation  $\psi^{\bullet\bullet} + \psi'' = -\Upsilon_4$ ;
2. for a nonzero constant  $h_0^2$ , and given  $\Upsilon_2$ ,

$$\varsigma(r, \vartheta, \varphi, \theta) = \varsigma_{[0]}(r, \vartheta, \theta) + h_0^2 \int \Upsilon_2 f^* [f - f_0] d\varphi;$$

3. the N-connection coefficients are

$$\begin{aligned} w_i(r, \vartheta, \varphi, \theta) &= -\partial_i \varsigma / \varsigma^*, \\ n_k(r, \vartheta, \varphi, \theta) &= {}^1 n_k(r, \vartheta, \theta) + {}^2 n_k(r, \vartheta, \theta) \int \frac{[f^*]^2 \varsigma}{[f - f_0]^3} d\varphi. \end{aligned}$$

The above presented class of metrics describes nonholonomic deformations of the Schwarzschild metric into (pseudo) Finsler configurations induced by the noncommutative parameter. Subjecting the coefficients of (1.83) to additional constraints of type (1.79) with nonzero sources  $\Upsilon_\alpha$ , we extract a subclass of solutions for noncommutative gravity with effective Levi-Civita connection.

**Noncommutative ellipsoidal deformations:** In this section, we provide a method of extracting ellipsoidal configurations from a general metric (1.83) with coefficients constrained to generate solutions on the Einstein equations for the canonical d-connection or Levi-Civita connection.

We consider a diagonal metric depending on noncommutative parameter  $\theta$  (in general, such a metric may not solve any gravitational field equations)

$${}^\theta \mathbf{g} = -d\xi \otimes d\xi - r^2(\xi) d\vartheta \otimes d\vartheta - r^2(\xi) \sin^2 \vartheta d\varphi \otimes d\varphi + \varpi^2(\xi) dt \otimes dt, \quad (1.84)$$

where the local coordinates and nontrivial coefficients of metric are

$$\begin{aligned} x^1 &= \xi, x^2 = \vartheta, y^3 = \varphi, y^4 = t, \\ \check{g}_1 &= -1, \check{g}_2 = -r^2(\xi), \check{h}_3 = -r^2(\xi) \sin^2 \vartheta, \check{h}_4 = \varpi^2(\xi), \end{aligned} \quad (1.85)$$

for  $\xi = \int dr \left| 1 - \frac{2\mu_0}{r} + \frac{\theta}{r^2} \right|^{1/2}$  and  $\varpi^2(r) = 1 - \frac{2\mu_0}{r} + \frac{\theta}{r^2}$ . For  $\theta = 0$  and variable  $\xi(r)$ , this metric is just the Schwarzschild solution written in spacetime spherical coordinates  $(r, \vartheta, \varphi, t)$ .

Target metrics are generated by nonholonomic deforms with  $g_i = \eta_i \check{g}_i$  and  $h_a = \eta_a \check{h}_a$  and some nontrivial  $w_i, n_i$ , where  $(\check{g}_i, \check{h}_a)$  are given by data

(1.85) and parametrized by an ansatz of type (1.83),

$$\begin{aligned}
{}^\theta_\eta \mathbf{g} &= -\eta_1(\xi, \vartheta, \theta) d\xi \otimes d\xi - \eta_2(\xi, \vartheta, \theta) r^2(\xi) d\vartheta \otimes d\vartheta \\
&\quad - \eta_3(\xi, \vartheta, \varphi, \theta) r^2(\xi) \sin^2 \vartheta \delta\varphi \otimes \delta\varphi + \eta_4(\xi, \vartheta, \varphi, \theta) \varpi^2(\xi) \delta t \otimes \delta t, \\
\delta\varphi &= d\varphi + w_1(\xi, \vartheta, \varphi, \theta) d\xi + w_2(\xi, \vartheta, \varphi, \theta) d\vartheta, \\
\delta t &= dt + n_1(\xi, \vartheta, \theta) d\xi + n_2(\xi, \vartheta, \theta) d\vartheta;
\end{aligned} \tag{1.86}$$

the coefficients of such metrics are constrained to be solutions of the system of equations (1.74). Such equations for  $\Upsilon_2 = 0$  state certain relations between the coefficients of the vertical metric and polarization functions,

$$h_3 = -h_0^2(b^*)^2 = \eta_3(\xi, \vartheta, \varphi, \theta) r^2(\xi) \sin^2 \vartheta, \quad h_4 = b^2 = \eta_4(\xi, \vartheta, \varphi, \theta) \varpi^2(\xi), \tag{1.87}$$

for  $|\eta_3| = (h_0)^2 |\check{h}_4 / \check{h}_3| \left[ \left( \sqrt{|\eta_4|} \right)^* \right]^2$ . In these formulas, we have to chose  $h_0 = \text{const}$  (it must be  $h_0 = 2$  in order to satisfy the condition (1.79)), where  $\eta_4$  can be any function satisfying the condition  $\eta_4^* \neq 0$ . We generate a class of solutions for any function  $b(\xi, \vartheta, \varphi, \theta)$  with  $b^* \neq 0$ . For classes of solutions with nontrivial sources, it is more convenient to work directly with  $\eta_4$ , for  $\eta_4^* \neq 0$  but, for vacuum configurations, we can chose as a generating function, for instance,  $h_4$ , for  $h_4^* \neq 0$ .

It is possible to compute the polarizations  $\eta_1$  and  $\eta_2$ , when  $\eta_1 = \eta_2 r^2 = e^{\psi(\xi, \vartheta)}$ , from (1.74) with  $\Upsilon_4 = 0$ , i.e. from  $\psi^{\bullet\bullet} + \psi'' = 0$ .

Putting the above defined values of coefficients in the ansatz (1.86), we find a class of exact vacuum solutions of the Einstein equations defining stationary nonholonomic deformations of the Schwarzschild metric,

$$\begin{aligned}
{}^\varepsilon \mathbf{g} &= -e^{\psi(\xi, \vartheta, \theta)} (d\xi \otimes d\xi + d\vartheta \otimes d\vartheta) \\
&\quad - 4 \left[ \left( \sqrt{|\eta_4(\xi, \vartheta, \varphi, \theta)|} \right)^* \right]^2 \varpi^2(\xi) \delta\varphi \otimes \delta\varphi + \eta_4(\xi, \vartheta, \varphi, \theta) \varpi^2(\xi) \delta t \otimes \delta t, \\
\delta\varphi &= d\varphi + w_1(\xi, \vartheta, \varphi, \theta) d\xi + w_2(\xi, \vartheta, \varphi, \theta) d\vartheta, \\
\delta t &= dt + {}^1n_1(\xi, \vartheta, \theta) d\xi + {}^1n_2(\xi, \vartheta, \theta) d\vartheta.
\end{aligned} \tag{1.88}$$

The N-connection coefficients  $w_i$  and  ${}^1n_i$  in (1.88) must satisfy the last two conditions from (1.79) in order to get vacuum metrics in Einstein gravity. Such vacuum solutions are for nonholonomic deformations of a static black hole metric into (non) holonomic noncommutative Einstein spaces with locally anistoropic backgrounds (on coordinate  $\varphi$ ) defined by an arbitrary function  $\eta_4(\xi, \vartheta, \varphi, \theta)$  with  $\partial_\varphi \eta_4 \neq 0$ , an arbitrary  $\psi(\xi, \vartheta, \theta)$  solving the 2-d Laplace equation and certain integration functions  ${}^1w_i(\xi, \vartheta, \varphi, \theta)$  and  ${}^1n_i(\xi, \vartheta, \theta)$ . The nonholonomic structure of such spaces depends parametrically on noncommutative parameter(s)  $\theta$ .



In general, the solutions from the target set of metrics (1.86), or (1.88), do not define black holes and do not describe obvious physical situations. Nevertheless, they preserve the singular character of the coefficient  $\varpi^2(\xi)$  vanishing on the horizon of a Schwarzschild black hole if we take only smooth integration functions for some small noncommutative parameters  $\theta$ . We can also consider a prescribed physical situation when, for instance,  $\eta_4$  mimics 3-d, or 2-d, solitonic polarizations on coordinates  $\xi, \vartheta, \varphi$ , or on  $\xi, \varphi$ .

### Extracting black hole and rotoid configurations

From a class of metrics (1.88) defining nonholonomic noncommutative deformations of the Schwarzschild solution depending on parameter  $\theta$ , it is possible to select locally anisotropic configurations with possible physical interpretation of gravitational vacuum configurations with spherical and/or rotoid (ellipsoid) symmetry.

**Linear parametric noncommutative polarizations:** Let us consider generating functions of type  $b^2 = q(\xi, \vartheta, \varphi) + \bar{\theta}s(\xi, \vartheta, \varphi)$  and, for simplicity, restrict our analysis only with linear decompositions on a small dimensionless parameter  $\bar{\theta} \sim \theta$ , with  $0 < \bar{\theta} \ll 1$ . This way, we shall construct off-diagonal exact solutions of the Einstein equations depending on  $\bar{\theta}$  which for rotoid configurations can be considered as a small eccentricity.<sup>34</sup> For  $b$ , we get  $(b^*)^2 = \left[ (\sqrt{|q|})^* \right]^2 \left[ 1 + \bar{\theta} \frac{1}{(\sqrt{|q|})^*} \left( \frac{s}{\sqrt{|q|}} \right)^* \right]$ , which allows us to compute the vertical coefficients of d-metric (1.88) (i.e  $h_3$  and  $h_4$  and corresponding polarizations  $\eta_3$  and  $\eta_4$ ) using formulas (1.87). One should emphasize that nonholonomic deformations are not obligatory related to noncommutative ones. For instance, in a particular case, we can generate nonholonomic deformations of the Schwarzschild solution not depending on  $\bar{\theta}$ : we have to put  $\bar{\theta} = 0$  in the above formulas and consider  $b^2 = q$  and  $(b^*)^2 = [(\sqrt{|q|})^*]^2$ .

Nonholonomic deforms to rotoid configurations can be generated for

$$q = 1 - \frac{2\mu(\xi, \vartheta, \varphi)}{r} \text{ and } s = \frac{q_0(r)}{4\mu^2} \sin(\omega_0\varphi + \varphi_0), \quad (1.89)$$

with  $\mu(\xi, \vartheta, \varphi) = \mu_0 + \bar{\theta}\mu_1(\xi, \vartheta, \varphi)$  (anisotropically polarized mass) with certain constants  $\mu, \omega_0$  and  $\varphi_0$  and arbitrary functions/polarizations  $\mu_1(\xi, \vartheta, \varphi)$  and  $q_0(r)$  to be determined from some boundary conditions, with  $\bar{\theta}$  treated

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<sup>34</sup>From a formal point of view, we can summarize on all orders  $(\bar{\theta})^2, (\bar{\theta})^3 \dots$  stating such recurrent formulas for coefficients when get convergent series to some functions depending both on spacetime coordinates and a parameter  $\bar{\theta}$ .

as the eccentricity of an ellipsoid.<sup>35</sup> Such a noncommutative nonholonomic configuration determines a small deformation of the Schwarzschild spherical horizon into a rotoid configuration with eccentricity  $\bar{\theta}$ .

We provide the general solution for noncommutative ellipsoidal black holes determined by nonholonomic h-components of metric and N-connection coefficients which "survive" in the limit  $\bar{\theta} \rightarrow 0$ , i.e. such values do not depend on noncommutative parameter. Dependence on noncommutativity is contained in v-components of metric. This class of stationary rotoid type solutions is parametrized in the form

$$\begin{aligned} {}^{rot}_{\theta} \mathbf{g} &= -e^{\psi} (d\xi \otimes d\xi + d\vartheta \otimes d\vartheta) - 4 \left[ (\sqrt{|q|})^* \right]^2 \left[ 1 + \bar{\theta} \frac{1}{(\sqrt{|q|})^*} \left( \frac{s}{\sqrt{|q|}} \right)^* \right] \\ &\quad \delta\varphi \otimes \delta\varphi + (q + \bar{\theta}s) \delta t \otimes \delta t, \\ \delta\varphi &= d\varphi + w_1 d\xi + w_2 d\vartheta, \quad \delta t = dt + {}^1n_1 d\xi + {}^1n_2 d\vartheta, \end{aligned}$$

with functions  $q(\xi, \vartheta, \varphi)$  and  $s(\xi, \vartheta, \varphi)$  given by formulas (1.89) and N-connection coefficients  $w_i(\xi, \vartheta, \varphi)$  and  $n_i = {}^1n_i(\xi, \vartheta)$  subjected to conditions  $w_1 w_2 \left( \ln \left| \frac{w_1}{w_2} \right| \right)^* = w_2^\bullet - w_1'$ ,  $w_i^* \neq 0$ ; or  $w_2^\bullet - w_1' = 0$ ,  $w_i^* = 0$ ;  ${}^1n_1'(\xi, \vartheta) - {}^1n_2^\bullet(\xi, \vartheta) = 0$  and  $\psi(\xi, \vartheta)$  being any function for which  $\psi^{\bullet\bullet} + \psi'' = 0$ .

**Rotoids and noncommutative solitonic distributions:** There are static three dimensional solitonic distributions  $\eta(\xi, \vartheta, \varphi, \theta)$ , defined as solutions of a solitonic equation<sup>36</sup>  $\eta^{\bullet\bullet} + \epsilon(\eta' + 6\eta \eta^* + \eta^{***})^* = 0$ ,  $\epsilon = \pm 1$ , resulting in stationary black ellipsoid-solitonic noncommutative spacetimes  ${}^{\theta}\mathbf{V}$  generated as further deformations of a metric  ${}^{rot}_{\theta} \mathbf{g}$  (1.90). Such metrics are of type

$$\begin{aligned} {}^{rot}_{sol\theta} \mathbf{g} &= -e^{\psi} (d\xi \otimes d\xi + d\vartheta \otimes d\vartheta) \\ &\quad - 4 \left[ (\sqrt{|\eta q|})^* \right]^2 \left[ 1 + \bar{\theta} \frac{1}{(\sqrt{|\eta q|})^*} \left( \frac{s}{\sqrt{|\eta q|}} \right)^* \right] \delta\varphi \otimes \delta\varphi \\ &\quad + \eta (q + \bar{\theta}s) \delta t \otimes \delta t, \\ \delta\varphi &= d\varphi + w_1 d\xi + w_2 d\vartheta, \quad \delta t = dt + {}^1n_1 d\xi + {}^1n_2 d\vartheta. \end{aligned} \tag{1.90}$$

For small values of  $\bar{\theta}$ , a possible spacetime noncommutativity determines nonholonomic embedding of the Schwarzschild solution into a solitonic vacuum. In the limit of small polarizations, when  $|\eta| \sim 1$ , it is preserved the

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<sup>35</sup>we can relate  $\bar{\theta}$  to an eccentricity because the coefficient  $h_4 = b^2 = \eta_4(\xi, \vartheta, \varphi, \bar{\theta}) \varpi^2(\xi)$  becomes zero for data (1.89) if  $r_+ \simeq 2\mu_0/[1 + \bar{\theta} \frac{q_0(r)}{4\mu^2} \sin(\omega_0\varphi + \varphi_0)]$ , which is the "parametric" equation for an ellipse  $r_+(\varphi)$  for any fixed values  $\frac{q_0(r)}{4\mu^2}, \omega_0, \varphi_0$  and  $\mu_0$

<sup>36</sup>a function  $\eta$  can be a solution of any three dimensional solitonic and/or other non-linear wave equations

black hole character of metrics and the solitonic distribution can be considered as on a Schwarzschild background. It is also possible to take such parameters of  $\eta$  when a black hole is nonholonomically placed on a "gravitational hill" defined by a soliton induced by spacetime noncommutativity.

A vacuum metric (1.90) can be generalized for (pseudo) Finsler spaces with canonical d-connection as a solution of equations  $\hat{\mathbf{R}}_{\alpha\beta} = 0$  (1.70) if the metric is generalized to a subclass of (1.86) with stationary coefficients subjected to conditions

$$\begin{aligned} \psi^{\bullet\bullet}(\xi, \vartheta, \bar{\theta}) + \psi''(\xi, \vartheta, \bar{\theta}) &= 0; \\ h_3 &= \pm e^{-2\phi} \frac{(h_4^*)^2}{h_4} \text{ for given } h_4(\xi, \vartheta, \varphi, \bar{\theta}), \phi = {}^0\phi = \text{const}; \\ w_i &= w_i(\xi, \vartheta, \varphi, \bar{\theta}) \text{ are any functions}; \\ n_i &= {}^1n_i(\xi, \vartheta, \bar{\theta}) + {}^2n_i(\xi, \vartheta, \bar{\theta}) \int (h_4^*)^2 |h_4|^{-5/2} dv, \quad n_i^* \neq 0; \\ &= {}^1n_i(\xi, \vartheta, \bar{\theta}), n_i^* = 0, \end{aligned}$$

for  $h_4 = \eta(\xi, \vartheta, \varphi, \bar{\theta}) [q(\xi, \vartheta, \varphi) + \bar{\theta}s(\xi, \vartheta, \varphi)]$ . In the limit  $\bar{\theta} \rightarrow 0$ , we get a Schwarzschild configuration mapped nonholonomically on a N-anholonomic (pseudo) Riemannian spacetime with a prescribed nontrivial N-connection structure.

### Noncommutative gravity and (pseudo) Finsler variables

We summarize the main steps of such noncommutative complex Finsler – (pseudo) Riemannian transform:

1. Let us consider a solution for (non)holonomic noncommutative generalized Einstein gravity with a metric<sup>37</sup>

$$\begin{aligned} {}^\theta \mathring{\mathbf{g}} &= \mathring{g}_i dx^i \otimes dx^i + \mathring{h}_a (dy^a + \mathring{N}_j^a dx^j) \otimes (dy^a + \mathring{N}_i^a dx^i) \\ &= \mathring{g}_i e^i \otimes e^i + \mathring{h}_a \mathring{\mathbf{e}}^a \otimes \mathring{\mathbf{e}}^a = \mathring{g}_{i''j''} e^{i''} \otimes e^{j''} + \mathring{h}_{a''b''} \mathring{\mathbf{e}}^{a''} \otimes \mathring{\mathbf{e}}^{b''} \end{aligned}$$

related to an arbitrary (pseudo) Riemannian metric with transforms of type  ${}^\theta \mathring{\mathbf{g}}_{\alpha''\beta''} = \mathring{\mathbf{e}}_{\alpha'}^{\alpha''} \mathring{\mathbf{e}}_{\beta'}^{\beta''} {}^\theta \mathbf{g}_{\alpha'\beta'}$  parametrized in the form

$$\mathring{g}_{i''j''} = g_{i'j'} \mathring{\mathbf{e}}_{i''}^{i'} \mathring{\mathbf{e}}_{j''}^{j'} + h_{a'b'} \mathring{\mathbf{e}}_{i''}^{a'} \mathring{\mathbf{e}}_{j''}^{b'}, \quad \mathring{h}_{a''b''} = g_{i'j'} \mathring{\mathbf{e}}_{a''}^{i'} \mathring{\mathbf{e}}_{b''}^{j'} + h_{a'b'} \mathring{\mathbf{e}}_{a''}^{a'} \mathring{\mathbf{e}}_{b''}^{b'}.$$

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<sup>37</sup>we shall omit the left label  $\theta$  in this section if this will not result in ambiguities

For  $\mathring{\mathbf{e}}_{i''}^{i'} = \delta_{i''}^{i'}$ ,  $\mathring{\mathbf{e}}_{a''}^{a'} = \delta_{a''}^{a'}$ , we write  $\mathring{g}_{i''} = g_{i''} + h_{a'} \left( \mathring{\mathbf{e}}_{i''}^{a'} \right)^2$ ,  $\mathring{h}_{a''} = g_{i'} \left( \mathring{\mathbf{e}}_{a''}^{i'} \right)^2 + h_{a''}$ , i.e. in a form of four equations for eight unknown variables  $\mathring{\mathbf{e}}_{i''}^{a'}$  and  $\mathring{\mathbf{e}}_{a''}^{i'}$ , and  $\mathring{N}_{i''}^{a''} = \mathring{\mathbf{e}}_{i''}^{i'} \mathring{\mathbf{e}}_{a''}^{a'}$ ,  $N_{i'}^{a'} = N_{i''}^{a''}$ .

2. We choose on  ${}^\theta \mathbf{V}$  a fundamental Finsler function  $F = {}^3F(x^i, v, \theta) + {}^4F(x^i, y, \theta)$  inducing canonically a d-metric of type

$$\begin{aligned} {}^\theta \mathbf{f} &= f_i dx^i \otimes dx^i + f_a (dy^a + {}^c N_j^a dx^j) \otimes (dy^a + {}^c N_i^a dx^i), \\ &= f_i e^i \otimes e^i + f_a {}^c \mathbf{e}^a \otimes {}^c \mathbf{e}^a \end{aligned}$$

determined by data  ${}^\theta \mathbf{f}_{\alpha\beta} = \left[ f_i, f_a, {}^c N_j^a \right]$  in a canonical N-elongated base  ${}^c \mathbf{e}^\alpha = (dx^i, {}^c \mathbf{e}^a = dy^a + {}^c N_i^a dx^i)$ .

3. We define  $g_{i'} = f_{i'} \left( \frac{\dot{w}_{i'}}{c w_{i'}} \right)^2 \frac{h_{3'}}{f_{3'}}$  and  $g_{i'} = f_{i'} \left( \frac{\dot{n}_{i'}}{c n_{i'}} \right)^2 \frac{h_{4'}}{f_{4'}}$ . Both formulas are compatible if  $\dot{w}_{i'}$  and  $\dot{n}_{i'}$  are constrained to satisfy the conditions  $\Theta_{1'} = \Theta_{2'} = \Theta$ , where  $\Theta_{i'} = \left( \frac{\dot{w}_{i'}}{c w_{i'}} \right)^2 \left( \frac{\dot{n}_{i'}}{c n_{i'}} \right)^2$ , and  $\Theta = \left( \frac{\dot{w}_{1'}}{c w_{1'}} \right)^2 \left( \frac{\dot{n}_{1'}}{c n_{1'}} \right)^2 = \left( \frac{\dot{w}_{2'}}{c w_{2'}} \right)^2 \left( \frac{\dot{n}_{2'}}{c n_{2'}} \right)^2$ . Using  $\Theta$ , we compute  $g_{i'} = \left( \frac{\dot{w}_{i'}}{c w_{i'}} \right)^2 \frac{f_{i'}}{f_{3'}}$  and  $h_{3'} = h_{4'} \Theta$ , where (in this case) there is not summing on indices. So, we constructed the data  $g_{i'}, h_{a'}$  and  $w_{i'}, n_{j'}$ .
4. The values  $\mathring{\mathbf{e}}_{i''}^{a'}$  and  $\mathring{\mathbf{e}}_{a''}^{i'}$  are determined as any nontrivial solutions of

$$\mathring{g}_{i''} = g_{i''} + h_{a'} \left( \mathring{\mathbf{e}}_{i''}^{a'} \right)^2, \mathring{h}_{a''} = g_{i'} \left( \mathring{\mathbf{e}}_{a''}^{i'} \right)^2 + h_{a''}, \mathring{N}_{i''}^{a''} = N_{i''}^{a''}.$$

For instance, we can choose and, respectively, express

$$\begin{aligned} \mathring{\mathbf{e}}_{1''}^{3'} &= \pm \sqrt{|(\mathring{g}_{1''} - g_{1''}) / h_{3'}|}, \mathring{\mathbf{e}}_{2''}^{3'} = 0, \mathring{\mathbf{e}}_{i''}^{4'} = 0 \\ \mathring{\mathbf{e}}_{a''}^{1'} &= 0, \mathring{\mathbf{e}}_{3''}^{2'} = 0, \mathring{\mathbf{e}}_{4''}^{2'} = \pm \sqrt{|(\mathring{h}_{4''} - h_{4''}) / g_{2'}|}, \end{aligned}$$

$$\text{and } e_{1'}^{1'} = \pm \sqrt{\left| \frac{f_1}{g_{1'}} \right|}, e_{2'}^{2'} = \pm \sqrt{\left| \frac{f_2}{g_{2'}} \right|}, e_{3'}^{3'} = \pm \sqrt{\left| \frac{f_3}{h_{3'}} \right|}, e_{4'}^{4'} = \pm \sqrt{\left| \frac{f_4}{h_{4'}} \right|}.$$

Finally, in this section, we conclude that any model of noncommutative nonholonomic gravity with distributions of type (1.63) and/or (1.98) can be equivalently re-formulated as a Finsler gravity induced by a generating function of type  $F = {}^3F + {}^4F$ . In the limit  $\theta \rightarrow 0$ , for any solution  ${}^\theta \mathring{\mathbf{g}}$ , there is a scheme of two nonholonomic transforms which allows us to rewrite the Schwarzschild solution and its noncommutative/nonholonomic deformations as a Finsler metric  ${}^\theta \mathbf{f}$ .

### 1.2.7 Geometric methods and quantum gravity

Let us consider a real (pseudo) Riemann manifold  $V^{2n}$  of necessary smooth class;  $\dim V^{2n} = 2n$ , where the dimension  $n \geq 2$  is fixed.<sup>38</sup> We label the local coordinates in the form  $u^\alpha = (x^i, y^a)$ , or  $u = (x, y)$ , where indices run values  $i, j, \dots = 1, 2, \dots, n$  and  $a, b, \dots = n+1, n+2, \dots, n+n$ , and  $x^i$  and  $y^a$  are respectively the conventional horizontal / holonomic (h) and vertical / nonholonomic coordinates (v). For the local Euclidean signature, we consider that all local basis vectors are real but, for the pseudo-Euclidean signature  $(-, +, +, +)$ , we introduce  $e_{j=1} = i\partial/\partial x^1$ , where  $i$  is the imaginary unity,  $i^2 = -1$ , and the local coordinate basis vectors can be written in the form  $e_\alpha = \partial/\partial u^\alpha = (i\partial/\partial x^1, \partial/\partial x^2, \dots, \partial/\partial x^n, \partial/\partial y^a)$ .<sup>39</sup> The Einstein's rule on summing up/low indices will be applied unless indicated otherwise.

Any metric on  $V^{2n}$  can be written as

$$\mathbf{g} = g_{ij}(x, y) e^i \otimes e^j + h_{ab}(x, y) e^a \otimes e^b, \quad (1.91)$$

where the dual vielbeins (tetrads, in four dimensions)  $e^a = (e^i, e^a)$  are parametrized  $e^i = e^i_{\underline{i}}(u)dx^{\underline{i}}$  and  $e^a = e^a_{\underline{a}}(u)dx^{\underline{a}} + e^a_{\underline{a}}(u)dy^{\underline{a}}$ , for  $e_{\underline{a}} = \partial/\partial u^{\underline{a}} = (e_{\underline{i}} = \partial/\partial x^{\underline{i}}, e_{\underline{a}} = \partial/\partial y^{\underline{a}})$  and  $e^{\underline{b}} = du^{\underline{b}} = (e^{\underline{j}} = dx^{\underline{j}}, dy^{\underline{b}})$  being, respectively, any fixed local coordinate base and dual base.

**Proposition 1.2.4** *Any metric  $\mathbf{g}$  (1.91) can be expressed in the form*

$$\check{\mathbf{g}} = \check{g}_{i'j'}(x, y) \check{e}^{i'} \otimes \check{e}^{j'} + \check{h}_{a'b'}(x, y) \check{e}^{a'} \otimes \check{e}^{b'}, \quad (1.92)$$

where  $\check{e}^{i'} = \delta_{\underline{i}}^{i'} dx^{\underline{i}}$  and  $\check{e}^{a'} = \delta_{\underline{a}}^{a'} dy^{\underline{a}} + \check{N}_{\underline{i}}^{a'}(u)dx^{\underline{i}}$  for

$$\check{h}_{a'b'}(u) = \frac{1}{2} \frac{\partial^2 \mathcal{L}(x^{i'}, y^{c'})}{\partial y^{a'} \partial y^{b'}}, \quad (1.93)$$

$$\check{N}_{\underline{i}}^{a'}(u) = \frac{\partial G^{a'}(x, y)}{\partial y^{n+\underline{i}}}, \quad (1.94)$$

where  $\delta_{\underline{i}}^{i'}$  is the Kronecker symbol,  $\check{g}_{i'j'} = \check{h}_{n+i' n+j'}$  and  $\check{h}^{ab}$  is the inverse of  $\check{h}_{a'b'}$ , for  $\det |\check{h}_{a'b'}| \neq 0$  and

$$2G^{a'}(x, y) = \frac{1}{2} \check{h}^{a' n+i} \left( \frac{\partial^2 \mathcal{L}}{\partial y^i \partial x^k} y^{n+k} - \frac{\partial \mathcal{L}}{\partial x^i} \right), \quad (1.95)$$

where  $i, k = 1, 2, \dots, n$ .

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<sup>38</sup>for constructions related to Einstein's gravity  $2n = 4$

<sup>39</sup>for simplicity, we shall omit to write in explicit form the imaginary unity considering that we can always distinguish the pseudo-Euclidean signature by a corresponding metric form or a local system of coordinates with a coordinate proportional to the imaginary unit

By a straightforward computation, we can prove

**Lemma 1.2.3** *Considering  $\mathcal{L}$  from (1.93) and (1.95) to be a regular Lagrangian, we have that the Euler–Lagrange equations*

$$\frac{d}{d\tau} \left( \frac{\partial \mathcal{L}}{\partial y^i} \right) - \frac{\partial \mathcal{L}}{\partial x^i} = 0, \quad (1.96)$$

where  $y^i = y^{n+i} = \frac{dx^i}{d\tau}$ , for  $x^i(\tau)$  depending on the parameter  $\tau$ . These equations are equivalent to the “nonlinear” geodesic equations

$$\frac{d^2 x^i}{d\tau^2} + 2G^i(x^k, \frac{dx^j}{d\tau}) = 0 \quad (1.97)$$

defining the paths of a canonical semispray  $S = y^i \frac{\partial}{\partial x^i} - 2G^a(x, y) \frac{\partial}{\partial y^a}$ , for  $G^a$  given by equations (1.95).

The Lemma motivates

**Definition 1.2.12** *A (pseudo) Riemannian space with metric  $\mathbf{g}$  (1.91) is modelled by a mechanical system with regular effective Lagrangian  $\mathcal{L}$  if there is a nontrivial frame transform defined by any  $e^i_{\underline{i}}, e^a_{\underline{i}}$  and  $e^a_{\underline{a}}$  when  $\mathbf{g} = \check{\mathbf{g}}$  (1.92).*

Inversely, we say that a regular mechanical model with Lagrangian  $\mathcal{L}$  and Euler–Lagrange equations (1.96) is geometrized in terms of a (pseudo) Riemannian geometry with metric  $\mathbf{g}$  (1.91) if  $\mathcal{L}$  is a generating function for (1.93), (1.95) and (1.94), when  $\mathbf{g} = \check{\mathbf{g}}$  (1.92) and the nonlinear geodesic equations (1.97) are equivalent to (1.96).

Any equivalent modelling of regular mechanical systems as (pseudo) Riemannian spaces introduces additional geometric structures on  $V^{2n}$ .

**Definition 1.2.13** *A nonlinear connection ( $N$ -connection)  $\mathbf{N}$  on  $V^{2n}$  is defined by a Whitney sum (nonholonomic distribution)*

$$T(V^{2n}) = h(V^{2n}) \oplus v(V^{2n}), \quad (1.98)$$

splitting globally the tangent bundle  $T(V^{2n})$  into respective  $h$ - and  $v$ -subspaces,  $h(V^{2n})$  and  $v(V^{2n})$ , given locally by a set of coefficients  $N^a_i(x, y)$  where  $\mathbf{N} = N^a_i(x, y) dx^i \otimes \frac{\partial}{\partial y^a}$ .

We note that a subclass of linear connections is defined by  $N^a_i = \Gamma^a_b(x) y^b$ .

We can perform  $N$ -adapted geometric constructions by defining the coefficients of geometric objects (and associated equations) with respect to  $N$ -adapted noholonomic frames of type (1.3) and (1.4). The  $N$ -adapted tensors, vectors, forms, etc., are called respectively distinguished tensors, etc., (in brief,  $d$ -tensors,  $d$ -vectors,  $d$ -forms, etc.). For instance, a vector field  $\mathbf{X} \in T\mathbf{V}^{2n}$  is expressed as  $\mathbf{X} = (hX, vX)$ , or  $\mathbf{X} = X^\alpha \mathbf{e}_\alpha = X^i \mathbf{e}_i + X^a e_a$ , where  $hX = X^i \mathbf{e}_i$  and  $vX = X^a e_a$  state, respectively, the horizontal (h) and vertical (v) components of the vector adapted to the  $N$ -connection structure.

**Proposition 1.2.5** *Any effective regular Lagrangian  $\mathcal{L}$ , prescribed on  $\mathbf{V}^{2n}$ , defines a canonical  $N$ -connection structure  $\tilde{\mathbf{N}} = \{\tilde{N}^{a'}_{\underline{i}}(u)\}$  (1.94) and preferred frame structures  $\tilde{\mathbf{e}}_\nu = (\tilde{\mathbf{e}}_i, e_{a'})$  and  $\tilde{\mathbf{e}}^\mu = (e^i, \tilde{\mathbf{e}}^{a'})$ .*

**Proof.** The proposition can be proved by straightforward computations. The coefficients  $\tilde{N}^{a'}_{\underline{i}}$  satisfy the conditions of Definition 1.2.13. We define  $\tilde{\mathbf{e}}_\nu = (\tilde{\mathbf{e}}_i, e_a)$  and  $\tilde{\mathbf{e}}^\mu = (e^i, \tilde{\mathbf{e}}^a)$  in explicit form by introducing  $\tilde{N}^{a'}_{\underline{i}}$ , respectively, in formulas (1.3) and (1.4).  $\square$

Similar constructions can be defined for  $\mathcal{L} = \mathcal{F}^2(x, y)$ , where an effective Finsler metric  $\mathcal{F}$  is a differentiable function of class  $C^\infty$  in any point  $(x, y)$  with  $y \neq 0$  and is continuous in any point  $(x, 0)$ ;  $\mathcal{F}(x, y) > 0$  if  $y \neq 0$ ; it satisfies the homogeneity condition  $\mathcal{F}(x, \beta y) = |\beta| \mathcal{F}(x, y)$  for any nonzero  $\beta \in \mathbb{R}$  and the Hessian (1.93) computed for  $\mathcal{L} = \mathcal{F}^2$  is positive definite. In this case, we can say that a (pseudo) Riemannian space with metric  $\mathbf{g}$  is modeled by an effective Finsler geometry and, inversely, a Finsler geometry is modeled on a (pseudo) Riemannian space.

**Definition 1.2.14** *A (pseudo) Riemannian manifold  $\mathbf{V}^{2n}$  is nonholonomic ( $N$ -anholonomic) if it is provided with a nonholonomic distribution on  $TV^{2n}$  ( $N$ -connection structure  $\mathbf{N}$ ).*

We formulate the first main result in this paper:

**Theorem 1.2.10** *Any (pseudo) Riemannian space can be transformed into a  $N$ -anholonomic manifold  $\mathbf{V}^{2n}$  modeling an effective Lagrange (or Finsler) geometry by prescribing a generating Lagrange (or Finsler) function  $\mathcal{L}(x, y)$  (or  $\mathcal{F}(x, y)$ ).*

**Proof.** Such a proof follows from Propositions 1.2.4 and 1.2.5 and Lemma 1.2.3. It should be noted that, by corresponding vielbein transforms  $e^i_{\underline{i}}, e^a_{\underline{a}}$  and  $e^a_{\underline{a}}$ , any metric  $\mathbf{g}$  with coefficients defined with respect

to an arbitrary co-frame  $\mathbf{e}^\mu$ , see (1.91), can be transformed into canonical Lagrange (Finsler) ones,  $\check{\mathbf{g}}$  (1.92). The  $\check{\mathbf{g}}$  coefficients are computed with respect to  $\check{\mathbf{e}}^\mu = (e^i, \check{\mathbf{e}}^a)$ , with the associated N-connection structure  $\check{N}_{\check{i}}^{a'}$ , all defined by a prescribed  $\mathcal{L}(x, y)$  (or  $\mathcal{F}(x, y)$ ).  $\square$

Finally, it should be noted that considering an arbitrary effective Lagrangian  $\mathcal{L}(x, y)$  on a four dimensional (pseudo) Riemannian spacetime and defining a corresponding  $2 + 2$  decomposition, local Lorentz invariance is not violated. We can work in any reference frame and coordinates, but the constructions adapted to the canonical N-connection structure and an analogous mechanical modeling are more convenient for developing a formalism of deformation quantization of gravity following the appropriate methods for Lagrange-Finsler and almost Kähler spaces.

### Almost Kähler Models for (Pseudo) Riemannian and Lagrange Spaces

The goal of this section is to prove that for any (pseudo) Riemannian metric and  $n + n$  splitting we can define canonical almost symplectic structures. The analogous mechanical modeling developed in previous sections is important from two points of view: Firstly, it provides both geometric and physical interpretations for the class of nonholonomic transforms with  $n + n$  splitting and adapting to the N-connection. Secondly, such canonical constructions can be equivalently redefined as a class of almost Kähler geometries with associated N-connection when certain symplectic forms and linear connection structures are canonically induced by the metric  $\mathbf{g}(x, y)$  and effective Lagrangian  $\mathcal{L}(x, y)$  on  $\mathbf{V}^{2n}$ .

Let  $\check{\mathbf{e}}_{\alpha'} = (\check{\mathbf{e}}_i, e_{b'})$  and  $\check{\mathbf{e}}^{\alpha'} = (e^i, \check{\mathbf{e}}^{b'})$  be defined respectively by (1.3) and (1.4) for the canonical N-connection  $\check{\mathbf{N}}$  stated by a metric structure  $\mathbf{g} = \check{\mathbf{g}}$  on  $\mathbf{V}^{2n}$ . We introduce a linear operator  $\check{\mathbf{J}}$  acting on tangent vectors to  $\mathbf{V}^{2n}$  following formulas  $\check{\mathbf{J}}(\check{\mathbf{e}}_i) = -e_{n+i}$  and  $\check{\mathbf{J}}(e_{n+i}) = \check{\mathbf{e}}_i$ , where the index  $a'$  runs values  $n + i$  for  $i = 1, 2, \dots, n$  and  $\check{\mathbf{J}} \circ \check{\mathbf{J}} = -\mathbf{I}$  for  $\mathbf{I}$  being the unity matrix. Equivalently, we introduce a tensor field on  $\mathbf{V}^{2n}$ ,

$$\begin{aligned} \check{\mathbf{J}} &= \check{\mathbf{J}}_{\beta}^{\alpha} e_{\alpha} \otimes e^{\beta} = \check{\mathbf{J}}_{\beta}^{\alpha} \frac{\partial}{\partial u^{\alpha}} \otimes du^{\beta} = \check{\mathbf{J}}_{\beta'}^{\alpha'} \check{\mathbf{e}}_{\alpha'} \otimes \check{\mathbf{e}}^{\beta'} = -e_{n+i} \otimes e^i + \check{\mathbf{e}}_i \otimes \check{\mathbf{e}}^{n+i} \\ &= -\frac{\partial}{\partial y^i} \otimes dx^i + \left( \frac{\partial}{\partial x^i} - \check{N}_i^{n+j} \frac{\partial}{\partial y^j} \right) \otimes (dy^i + \check{N}_k^{n+i} dx^k). \end{aligned}$$

It is clear that  $\check{\mathbf{J}}$  defines globally an almost complex structure on  $\mathbf{V}^{2n}$  completely determined by a fixed  $\mathcal{L}(x, y)$ .



**Definition 1.2.15** *The Nijenhuis tensor field for any almost complex structure  $\mathbf{J}$  determined by a  $N$ -connection (equivalently, the curvature of  $N$ -connection) is defined as*

$$\mathbf{J}\Omega(\mathbf{X}, \mathbf{Y}) = -[\mathbf{X}, \mathbf{Y}] + [\mathbf{JX}, \mathbf{JY}] - \mathbf{J}[\mathbf{JX}, \mathbf{Y}] - \mathbf{J}[\mathbf{X}, \mathbf{JY}], \quad (1.99)$$

for any  $d$ -vectors  $\mathbf{X}$  and  $\mathbf{Y}$ .

With respect to  $N$ -adapted bases the Neijenhuis tensor  $\mathbf{J}\Omega = \{\Omega_{ij}^a\}$  has the coefficients

$$\Omega_{ij}^a = \frac{\partial N_i^a}{\partial x^j} - \frac{\partial N_j^a}{\partial x^i} + N_i^b \frac{\partial N_j^a}{\partial y^b} - N_j^b \frac{\partial N_i^a}{\partial y^b}. \quad (1.100)$$

A  $N$ -anholonomic manifold  $\mathbf{V}^{2n}$  is integrable if  $\Omega_{ij}^a = 0$ . We get a complex structure if and only if both the  $h$ - and  $v$ -distributions are integrable, i.e., if and only if  $\Omega_{ij}^a = 0$  and  $\frac{\partial N_j^a}{\partial y^i} - \frac{\partial N_i^a}{\partial y^j} = 0$ .

**Definition 1.2.16** *An almost symplectic structure on a manifold  $V^{n+m}$ ,  $\dim V^{n+m} = n+m$ , is defined by a nondegenerate 2-form  $\theta = \frac{1}{2}\theta_{\alpha\beta}(u)e^\alpha \wedge e^\beta$ .*

We have

**Proposition 1.2.6** *For any  $\theta$  on  $V^{n+m}$ , there is a unique  $N$ -connection  $\mathbf{N} = \{N_i^a\}$  defined as a splitting  $TV^{n+m} = hV^{n+m} \oplus vV^{n+m}$ , where indices  $i, j, .. = 1, 2, ...n$  and  $a, b, ... = n+1, n+1, ...n+m$ . The function  $\theta$  satisfies the following conditions:*

$$\theta = (h\mathbf{X}, v\mathbf{Y}) = 0 \text{ and } \theta = h\theta + v\theta, \quad (1.101)$$

for any  $\mathbf{X} = h\mathbf{X} + v\mathbf{X}$ ,  $\mathbf{Y} = h\mathbf{Y} + v\mathbf{Y}$  and  $h\theta(\mathbf{X}, \mathbf{Y}) \doteq \theta(h\mathbf{X}, h\mathbf{Y})$ ,  $v\theta(\mathbf{X}, \mathbf{Y}) \doteq \theta(v\mathbf{X}, v\mathbf{Y})$ . Here the symbol " $\doteq$ " means "by definition".

**Proof.** For  $\mathbf{X} = \mathbf{e}_\alpha = (\mathbf{e}_i, e_a)$  and  $\mathbf{Y} = \mathbf{e}_\beta = (\mathbf{e}_l, e_b)$ , where  $\mathbf{e}_\alpha$  is a  $N$ -adapted basis of dimension  $n+m$ , we write the first equation in (1.101) as  $\theta = \theta(\mathbf{e}_i, e_a) = \theta(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^a}) - N_i^b \theta(\frac{\partial}{\partial y^b}, \frac{\partial}{\partial y^a}) = 0$ . We can find a unique solution form and define  $N_i^b$  if  $\text{rank}|\theta(\frac{\partial}{\partial y^b}, \frac{\partial}{\partial y^a})| = m$ . Denoting locally

$$\theta = \frac{1}{2}\theta_{ij}(u)e^i \wedge e^j + \frac{1}{2}\theta_{ab}(u)\mathbf{e}^a \wedge \mathbf{e}^b, \quad (1.102)$$

where the first term is for  $h\theta$  and the second term is  $v\theta$ , we get the second formula in (1.101). We may consider the particular case in which  $n = m$ .  $\square$

**Definition 1.2.17** An almost Hermitian model of a (pseudo) Riemannian space  $\mathbf{V}^{2n}$  equipped with an  $N$ -connection structure  $\mathbf{N}$  is defined by a triple  $\mathbf{H}^{2n} = (\mathbf{V}^{2n}, \theta, \mathbf{J})$ , where  $\theta(\mathbf{X}, \mathbf{Y}) \doteq \mathbf{g}(\mathbf{J}\mathbf{X}, \mathbf{Y})$ .

In addition, we have

**Definition 1.2.18** A space  $\mathbf{H}^{2n}$  is almost Kähler, denoted  $\mathbf{K}^{2n}$ , if and only if  $d\theta = 0$ .

If a (pseudo) Riemannian space is modeled by a Lagrange–Finsler geometry, the second main result of this paper follows

**Theorem 1.2.11** Having chosen a generating function  $\mathcal{L}(x, y)$  (or  $\mathcal{F}(x, y)$ ) on a (pseudo) Riemannian manifold  $V^{n+n}$ , we can model this space as an almost Kähler geometry, i.e.  $\check{\mathbf{H}}^{2n} = \check{\mathbf{K}}^{2n}$ .

**Proof.** For  $\mathbf{g} = \check{\mathbf{g}}$  (1.92) and structures  $\check{\mathbf{N}}$  and  $\check{\mathbf{J}}$  canonically defined by  $\mathcal{L}$ , we define  $\check{\theta}(\mathbf{X}, \mathbf{Y}) \doteq \check{\mathbf{J}}(\check{\mathbf{F}}\mathbf{X}, \mathbf{Y})$  for any  $d$ -vectors  $\mathbf{X}$  and  $\mathbf{Y}$ . In local  $N$ -adapted form form, we have

$$\begin{aligned} \check{\theta} &= \frac{1}{2} \check{\theta}_{\alpha\beta}(u) e^\alpha \wedge e^\beta = \frac{1}{2} \check{\theta}_{\underline{\alpha}\underline{\beta}}(u) du^\alpha \wedge du^\beta \\ &= \check{g}_{ij}(x, y) \check{e}^{n+i} \wedge dx^j = \check{g}_{ij}(x, y) (dy^{n+i} + \check{N}_k^{n+i} dx^k) \wedge dx^j. \end{aligned} \quad (1.103)$$

Let us consider the form  $\check{\omega} = \frac{1}{2} \frac{\partial \mathcal{L}}{\partial y^{n+i}} dx^i$ . A straightforward computation, using Proposition 1.2.5 and  $N$ -connection  $\check{\mathbf{N}}$  (1.94), shows that  $\check{\theta} = d\check{\omega}$ , which means that  $d\check{\theta} = dd\check{\omega} = 0$  and that the canonical effective Lagrange structures  $\mathbf{g} = \check{\mathbf{g}}$ ,  $\check{\mathbf{N}}$  and  $\check{\mathbf{J}}$  induce an almost Kähler geometry. Instead of "Lagrangian mechanics variables" we can introduce another type redefining  $\check{\theta}$  with respect to an arbitrary co-frame basis using vielbeins  $\mathbf{e}_{\underline{\alpha}}^\alpha$  and their duals  $\mathbf{e}_\alpha^{\underline{\alpha}}$ , defined by  $e^i_{\underline{i}}, e^a_{\underline{a}}$  and  $e^a_{\underline{a}}$ . So, we can compute  $\check{\theta}_{\alpha\beta} = \mathbf{e}_\alpha^{\underline{\alpha}} \mathbf{e}_\beta^{\underline{\beta}} \check{\theta}_{\underline{\alpha}\underline{\beta}}$  and express the 2-form (1.103) as  $\check{\theta} = \frac{1}{2} \check{\theta}_{ij}(u) e^i \wedge e^j + \frac{1}{2} \check{\theta}_{ab}(u) \check{\mathbf{e}}^a \wedge \check{\mathbf{e}}^b$ , see (1.102). The coefficients  $\check{\theta}_{ab} = \check{\theta}_{n+i \ n+j}$  above are equal, respectively, to the coefficients  $\check{\theta}_{ij}$  and the dual  $N$ -adapted basis  $\check{\mathbf{e}}^\alpha = (e^i, \check{\mathbf{e}}^a)$  is elongated by  $\check{N}_j^a$  (1.94). It should be noted that for a general 2-form  $\theta$  directly constructed from a metric  $\mathbf{g}$  and almost complex  $\mathbf{J}$  structures on  $V^{2n}$ , we have that  $d\theta \neq 0$ . For a  $n + n$  splitting induced by an effective Lagrange (Finsler) generating function, we have  $d\check{\theta} = 0$  which results in a canonical almost Kähler model completely defined by  $\mathbf{g} = \check{\mathbf{g}}$  and chosen  $\mathcal{L}(x, y)$  (or  $\mathcal{F}(x, y)$ ).  $\square$

**N-adapted symplectic connections:** In our approach, we work with nonholonomic (pseudo) Riemannian manifolds  $\mathbf{V}^{2n}$  enabled with an effective N-connection and almost symplectic structures defined canonically by the metric structure  $\mathbf{g} = \check{\mathbf{g}}$  and a fixed  $\mathcal{L}(x, y)$ . In this section, we analyze the class of linear connections that can be adapted to the N-connection and/or symplectic structure and defined canonically if a corresponding non-holonomic distribution is induced completely by  $\mathcal{L}$ , or  $\mathcal{F}$ .

From the class of arbitrary affine connections on  $\mathbf{V}^{2n}$ , one prefers to work with N-adapted linear connections, called distinguished connections (d-connections).

**Definition 1.2.19** *A linear connection on  $\mathbf{V}^{2n}$  is a d-connection*

$$\mathbf{D} = (hD; vD) = \{\Gamma_{\beta\gamma}^\alpha = (L_{jk}^i, {}^vL_{bk}^a; C_{jc}^i, {}^vC_{bc}^a)\},$$

*with local coefficients computed with respect to N-adapted (1.3) and (1.4), which preserves the distribution (1.98) under parallel transports.*

For a d-connection  $\mathbf{D}$ , we can define respectively the torsion and curvature tensors,

$$\mathbf{T}(\mathbf{X}, \mathbf{Y}) \doteq \mathbf{D}_\mathbf{X}\mathbf{Y} - \mathbf{D}_\mathbf{Y}\mathbf{X} - [\mathbf{X}, \mathbf{Y}], \quad (1.104)$$

$$\mathbf{R}(\mathbf{X}, \mathbf{Y})\mathbf{Z} \doteq \mathbf{D}_\mathbf{X}\mathbf{D}_\mathbf{Y}\mathbf{Z} - \mathbf{D}_\mathbf{Y}\mathbf{D}_\mathbf{X}\mathbf{Z} - \mathbf{D}_{[\mathbf{X}, \mathbf{Y}]}\mathbf{Z}, \quad (1.105)$$

where  $[\mathbf{X}, \mathbf{Y}] \doteq \mathbf{X}\mathbf{Y} - \mathbf{Y}\mathbf{X}$ , for any d-vectors  $\mathbf{X}$  and  $\mathbf{Y}$ . The coefficients  $\mathbf{T} = \{\mathbf{T}_{\beta\gamma}^\alpha\}$  and  $\mathbf{R} = \{\mathbf{R}_{\beta\gamma\tau}^\alpha\}$  can be written in terms of  $\mathbf{e}_\alpha$  and  $\mathbf{e}^\beta$  by introducing  $\mathbf{X} \rightarrow \mathbf{e}_\alpha$ ,  $\mathbf{Y} \rightarrow \mathbf{e}_\beta$ ,  $\mathbf{Z} \rightarrow \mathbf{e}_\gamma$  in (1.104) and (1.105).

**Definition 1.2.20** *A d-connection  $\mathbf{D}$  is metric compatible with a d-metric  $\mathbf{g}$  if  $\mathbf{D}_\mathbf{X}\mathbf{g} = 0$  for any d-vector field  $\mathbf{X}$ .*

If an almost symplectic structure is defined on a N-anholonomic manifold, one considers:

**Definition 1.2.21** *An almost symplectic d-connection  ${}_\theta\mathbf{D}$  on  $\mathbf{V}^{2n}$ , or (equivalently) a d-connection compatible with an almost symplectic structure  $\theta$ , is defined such that  ${}_\theta\mathbf{D}$  is N-adapted, i.e., it is a d-connection, and  ${}_\theta\mathbf{D}_\mathbf{X}\theta = 0$ , for any d-vector  $\mathbf{X}$ .*

We can always fix a d-connection  ${}_\circ\mathbf{D}$  on  $\mathbf{V}^{2n}$  and then construct an almost symplectic  ${}_\theta\mathbf{D}$ .

**Example 1.2.1** Let us represent  $\theta$  in  $N$ -adapted form (1.102). Having chosen  $a$

$$\begin{aligned}\circ\mathbf{D} &= \{h \circ D = (\circ D_k, \circ D_k); v \circ D = (\circ D_c, \circ D_c)\} \\ &= \{ \circ \Gamma_{\beta\gamma}^\alpha = (\circ L_{jk}^i, \circ L_{bk}^a; \circ C_{jc}^i, \circ C_{bc}^a) \},\end{aligned}$$

we can verify that

$$\begin{aligned}\theta\mathbf{D} &= \{\theta \theta D = (\theta D_k, \theta D_k); v \theta D = (\theta D_c, \theta D_c)\} \\ &= \{ \theta \Gamma_{\beta\gamma}^\alpha = (\theta L_{jk}^i, \theta L_{bk}^a; \theta C_{jc}^i, \theta C_{bc}^a) \},\end{aligned}$$

with

$$\begin{aligned}\theta L_{jk}^i &= \circ L_{jk}^i + \frac{1}{2}\theta^{ih} \circ D_k \theta_{jh}, \quad \theta L_{bk}^a = \circ L_{bk}^a + \frac{1}{2}\theta^{ae} \circ D_k \theta_{eb}, \\ \theta C_{jc}^i &= \circ C_{jc}^i + \frac{1}{2}\theta^{ih} \circ D_c \theta_{jh}, \quad \theta C_{bc}^a = \circ C_{bc}^a + \frac{1}{2}\theta^{ae} \circ D_c \theta_{eb},\end{aligned}\tag{1.106}$$

satisfies the conditions  $\theta D_k \theta_{jh} = 0$ ,  $\theta D_k \theta_{eb} = 0$ ,  $\theta D_c \theta_{jh} = 0$ ,  $\theta D_c \theta_{eb} = 0$ , which is equivalent to  $\theta \mathbf{D}_X \theta = 0$  from Definition 1.2.21.

Let us introduce the operators

$$\Theta_{jk}^{hi} = \frac{1}{2}(\delta_j^h \delta_k^i - \theta_{jk} \theta^{ih}) \text{ and } \Theta_{cd}^{ab} = \frac{1}{2}(\delta_c^a \delta_d^b - \theta_{cd} \theta^{ab}), \tag{1.107}$$

with the coefficients computed with respect to  $N$ -adapted bases. By straightforward computations, one proves the following theorem.

**Theorem 1.2.12** *The set of  $d$ -connections*

${}_s\Gamma_{\beta\gamma}^\alpha = ({}_sL_{jk}^i, {}_sL_{bk}^a; {}_sC_{jc}^i, {}_sC_{bc}^a)$  *which are compatible with an almost symplectic structure  $\theta$  (1.102), are parametrized by*

$$\begin{aligned}{}_sL_{jk}^i &= \theta L_{jk}^i + \Theta_{jl}^{hi} Y_{hk}^l, \quad {}_sL_{bk}^a = \theta L_{bk}^a + \Theta_{bd}^{ca} Y_{ck}^d, \\ {}_sC_{jc}^i &= \theta C_{jc}^i + \Theta_{jl}^{hi} Y_{hc}^l, \quad {}_sC_{bc}^a = \theta C_{bc}^a + \Theta_{bd}^{ca} Y_{ec}^d,\end{aligned}\tag{1.108}$$

where  $\theta \Gamma_{\beta\gamma}^\alpha = (\theta L_{jk}^i, \theta L_{bk}^a; \theta C_{jc}^i, \theta C_{bc}^a)$  is given by (1.106), the  $\Theta$ -operators are those from (1.107) and  $\mathbf{Y}_{\beta\gamma}^\alpha = (Y_{jk}^i, Y_{bk}^a, Y_{jc}^i, Y_{bc}^a)$  are arbitrary  $d$ -tensor fields.

From the set of metric and/or almost symplectic compatible  $d$ -connections on a (pseudo) Riemannian manifold  $V^{2n}$ , we can select those which are completely defined by  $\mathbf{g}$  and a prescribed effective Lagrange structure  $\mathcal{L}(x, y)$ :

**Theorem 1.2.13** *There is a unique normal d-connection*

$$\begin{aligned}\widehat{\mathbf{D}} &= \left\{ h\widehat{D} = (\widehat{D}_k, {}^v\widehat{D}_k = \widehat{D}_k); {}^v\widehat{D} = (\widehat{D}_c, {}^v\widehat{D}_c = \widehat{D}_c) \right\} \\ &= \left\{ \widehat{\Gamma}_{\beta\gamma}^\alpha = (\widehat{L}_{jk}^i, {}^v\widehat{L}_{n+j\ n+k}^{n+i} = \widehat{L}_{jk}^i; \widehat{C}_{jc}^i = {}^v\widehat{C}_{n+j\ c}^{n+i}, {}^v\widehat{C}_{bc}^a = \widehat{C}_{bc}^a) \right\},\end{aligned}\quad (1.109)$$

which is metric compatible,  $\widehat{D}_k\check{g}_{ij} = 0$  and  $\widehat{D}_c\check{g}_{ij} = 0$ , and completely defined by  $\mathbf{g} = \check{\mathbf{g}}$  and a fixed  $\mathcal{L}(x, y)$ .

**Proof.** First, we note that if a normal d-connection exists, it is completely defined by couples of h- and v-components  $\widehat{\mathbf{D}}_\alpha = (\widehat{D}_k, \widehat{D}_c)$ , i.e.  $\widehat{\Gamma}_{\beta\gamma}^\alpha = (\widehat{L}_{jk}^i, {}^v\widehat{C}_{bc}^a)$ . Choosing

$$\widehat{L}_{jk}^i = \frac{1}{2}\check{g}^{ih}(\check{\mathbf{e}}_k\check{g}_{jh} + \check{\mathbf{e}}_j\check{g}_{hk} - \check{\mathbf{e}}_h\check{g}_{jk}), \widehat{C}_{jk}^i = \frac{1}{2}\check{g}^{ih}\left(\frac{\partial\check{g}_{jh}}{\partial y^k} + \frac{\partial\check{g}_{hk}}{\partial y^j} - \frac{\partial\check{g}_{jk}}{\partial y^h}\right),\quad (1.110)$$

where  $\check{\mathbf{e}}_k = \partial/\partial x^k + \check{N}_k^a\partial/\partial y^a$ ,  $\check{N}_k^a$  and  $\check{g}_{jk} = \check{h}_{n+i\ n+j}$  are defined by canonical values (1.93) and (1.94) induced by a regular  $\mathcal{L}(x, y)$ , we can prove that this d-connection is unique and satisfies the conditions of the theorem. Using vielbeins  $\mathbf{e}_\alpha$  and their duals  $\mathbf{e}_\alpha^\alpha$ , defined by  $e^i_{\underline{i}}, e^a_{\underline{a}}$  and  $e^a_{\underline{a}}$ , we can compute the coefficients of  $\widehat{\Gamma}_{\beta\gamma}^\alpha$  (1.109) with respect to arbitrary frame basis  $e_\alpha$  and co-basis  $e^\alpha$  on  $V^{n+m}$ .  $\square$

Introducing the normal d-connection 1-form  $\widehat{\Gamma}_j^i = \widehat{L}_{jk}^i e^k + \widehat{C}_{jk}^i \check{\mathbf{e}}^k$ , for  $e^k = dx^k$  and  $\check{\mathbf{e}}^k = dy^k + \check{N}_i^k dx^i$ , we can prove that the Cartan structure equations are satisfied,

$$de^k - e^j \wedge \widehat{\Gamma}_j^k = -\widehat{\mathcal{T}}^i, \quad d\check{\mathbf{e}}^k - \check{\mathbf{e}}^j \wedge \widehat{\Gamma}_j^k = -{}^v\widehat{\mathcal{T}}^i, \quad (1.111)$$

and

$$d\widehat{\Gamma}_j^i - \widehat{\Gamma}_j^h \wedge \widehat{\Gamma}_h^i = -\widehat{\mathcal{R}}^i_j. \quad (1.112)$$

The h- and v-components of the torsion 2-form  $\widehat{\mathcal{T}}^\alpha = (\widehat{\mathcal{T}}^i, {}^v\widehat{\mathcal{T}}^i) = \widehat{\mathbf{T}}_{\tau\beta}^\alpha \check{\mathbf{e}}^\tau \wedge \check{\mathbf{e}}^\beta$  and from (1.111) the components are computed

$$\widehat{\mathcal{T}}^i = \widehat{C}_{jk}^i e^j \wedge \check{\mathbf{e}}^k, \quad {}^v\widehat{\mathcal{T}}^i = \frac{1}{2}\check{\Omega}_{kj}^i e^k \wedge e^j + \left(\frac{\partial\check{N}_k^i}{\partial y^j} - \widehat{L}_{kj}^i\right) e^k \wedge \check{\mathbf{e}}^j, \quad (1.113)$$

where  $\check{\Omega}_{kj}^i$  are coefficients of the curvature of the canonical N-connection  $\check{N}_k^i$  defined by formulas similar to (1.100). Such formulas also follow from (1.104) redefined for  $\widehat{\mathbf{D}}_\alpha$  and  $\check{\mathbf{e}}_\alpha$ , when the torsion  $\widehat{\mathbf{T}}_{\beta\gamma}^\alpha$  is parametrized as

$$\widehat{T}_{jk}^i = 0, \widehat{T}_{jc}^i = \widehat{C}_{jc}^i, \widehat{T}_{ij}^a = \check{\Omega}_{ij}^a, \widehat{T}_{ib}^a = e_b \check{N}_i^a - \widehat{L}_{bi}^a, \widehat{T}_{bc}^a = 0. \quad (1.114)$$

It should be noted that  $\hat{\mathbf{T}}$  vanishes on h- and v-subspaces, i.e.  $\hat{T}_{jk}^i = 0$  and  $\hat{T}_{bc}^a = 0$ , but certain nontrivial h-v-components induced by the nonholonomic structure are defined canonically by  $\mathbf{g} = \check{\mathbf{g}}$  and  $\mathcal{L}$ .

We can also compute the curvature 2-form from (1.112),

$$\hat{\mathcal{R}}^\tau_\gamma = \hat{\mathbf{R}}^\tau_{\gamma\alpha\beta} \check{\mathbf{e}}^\alpha \wedge \check{\mathbf{e}}^\beta = \frac{1}{2} \hat{R}^i_{jkh} e^k \wedge e^h + \hat{P}^i_{jka} e^k \wedge \check{\mathbf{e}}^a + \frac{1}{2} \hat{S}^i_{jcd} \check{\mathbf{e}}^c \wedge \check{\mathbf{e}}^d, \quad (1.115)$$

where the nontrivial N-adapted coefficients of curvature  $\hat{\mathbf{R}}^\alpha_{\beta\gamma\tau}$  of  $\hat{\mathbf{D}}$  are (such formulas can be proven also from (1.105) written for  $\hat{\mathbf{D}}_\alpha$  and  $\check{\mathbf{e}}_\alpha$ )

$$\begin{aligned} \hat{R}^i_{hjk} &= \check{\mathbf{e}}_k \hat{L}^i_{hj} - \check{\mathbf{e}}_j \hat{L}^i_{hk} + \hat{L}^m_{hj} \hat{L}^i_{mk} - \hat{L}^m_{hk} \hat{L}^i_{mj} - \hat{C}^i_{ha} \check{\mathcal{Q}}^a_{kj}, \\ \hat{P}^i_{jka} &= e_a \hat{L}^i_{jk} - \hat{\mathbf{D}}_k \hat{C}^i_{ja}, \quad \hat{S}^a_{bcd} = e_d \hat{C}^a_{bc} - e_c \hat{C}^a_{bd} + \hat{C}^e_{bc} \hat{C}^a_{ed} - \hat{C}^e_{bd} \hat{C}^a_{ec}. \end{aligned} \quad (1.116)$$

If instead of an effective Lagrange function, one considers a Finsler generating fundamental function  $\mathcal{F}^2$ , similar formulas for the torsion and curvature of the normal d-connection can also be found.

There is another very important property of the normal d-connection:

**Theorem 1.2.14** *The normal d-connection  $\hat{\mathbf{D}}$  defines a unique almost symplectic d-connection,  $\hat{\mathbf{D}} \equiv {}_\theta \hat{\mathbf{D}}$ , see Definition 1.2.21, which is N-adapted, i.e. it preserves under parallelism the splitting (1.98),  ${}_\theta \hat{\mathbf{D}} \mathbf{x} \check{\theta} = 0$  and  $\hat{T}^i_{jk} = \hat{T}^a_{bc} = 0$ , i.e. the torsion is of type (1.114).*

**Proof.** Applying the conditions of the theorem to the coefficients (1.110), the proof follows in a straightforward manner.  $\square$

In this section, we proved that a N-adapted and almost symplectic  $\hat{\mathbf{T}}^\alpha_{\beta\gamma}$  can be uniquely defined by a (pseudo) Riemannian metric  $\mathbf{g}$  if we prescribe an effective Lagrange, or Finsler, function  $\mathcal{L}$ , or  $\mathcal{F}$  on  $V^{2n}$ . This allows us to construct an analogous Lagrange model for gravity and, at the next step, to transform it equivalently in an almost Kähler structure adapted to a corresponding  $n + n$  spacetime splitting. For the Einstein metrics, we get a canonical  $2 + 2$  decomposition for which we can apply the Fedosov's quantization if the geometric objects and operators are adapted to the associated N-connection.

**Definition 1.2.22** *A (pseudo) Riemannian space is described in Lagrange-Finsler variables if its vielbein, metric and linear connection structures are equivalently transformed into corresponding canonical N-connection, Lagrange-Finsler metric and normal / almost symplectic d-connection structures.*

It should be noted that former approaches to the canonical and quantum loop quantization of gravity were elaborated for  $3 + 1$  fibrations and corresponding ADM and Ashtekar variables with further modifications. On the other hand, in order to elaborate certain approaches to deformation quantization of gravity, it is crucial to work with nonholonomic  $2 + 2$  structures, which is more convenient for certain Lagrange geometrized constructions and their almost symplectic variants. For other models, the  $3 + 1$  splitting preserves a number of similarities to Hamilton mechanics. In our approach, the spacetime decompositions are defined by corresponding N-connection structures, which can be induced canonically by effective Lagrange, or Finsler, generating functions. One works both with N-adapted metric coefficients and nonholonomic frame coefficients, the last ones being defined by generic off-diagonal metric coefficients and related N-connection coefficients. In the models related to  $3 + 1$  fibrations, one works with shift functions and frame variables which contain all dynamical information, instead of metrics.

We also discuss here the similarities and differences of preferred classes of linear connections used for  $3 + 1$  and  $2 + 2$  structures. In the first case, the Ashtekar variables (and further modifications) were proved to simplify the constraint structure of a gauge like theory to which the Einstein theory was transformed in order to develop a background independent quantization of gravity. In the second case, the analogs of Ashtekar variables are generated by a canonical Lagrange–Finsler type metric and/or corresponding almost symplectic structure, both adapted to the N-connection structure. It is also involved the normal d-connection which is compatible with the almost symplectic structure and completely defined by the metric structure, alternatively to the Levi–Civita connection (the last one is not adapted to the N-connection and induced almost symplectic structure). In fact, all constructions for the normal d-connection can be redefined in an equivalent form to the Levi–Civita connection, or in Ashtekar variables, but in such cases the canonical  $2 + 2$  splitting and almost Kähler structure are mixed by general frame and linear connection deformations.

### **Distinguished Fedosov’s Operators**

The Fedosov’s approach to deformation quantization will be extended for (pseudo) Riemannian manifolds  $V^{2n}$  endowed with an effective Lagrange function  $\mathcal{L}$ . The constructions elaborated by A. Karabegov and M. Schlichenmeier will be adapted to the canonical N-connection structure by considering decompositions with respect to  $\check{\mathbf{e}}_\nu = (\check{\mathbf{e}}_i, e_{a'})$  and  $\check{\mathbf{e}}^\mu = (e^i, \check{\mathbf{e}}^{a'})$  defined by a metric  $\mathbf{g}$  (1.91). For simplicity, we shall work only with the normal/

almost symplectic d-connection,  $\widehat{\mathbf{D}} \equiv {}_\theta \widehat{\mathbf{D}}$  (1.109), see Definition 1.2.21, but it should be emphasized here that we can use any d-connection from the family (1.108) and develop a corresponding deformation quantization. In this work, the formulas are redefined on nonholonomic (pseudo) Riemannian manifolds modeling effective regular mechanical systems and corresponding almost Kähler structures.

We introduce the tensor  $\check{\mathbf{A}}^{\alpha\beta} \doteq \check{\theta}^{\alpha\beta} - i \check{\mathbf{g}}^{\alpha\beta}$ , where  $\check{\theta}^{\alpha\beta}$  is the form (1.103) with "up" indices and  $\check{\mathbf{g}}^{\alpha\beta}$  is the inverse to  $\check{\mathbf{g}}_{\alpha\beta}$  stated by coefficients of (1.92). The local coordinates on  $\mathbf{V}^{2n}$  are parametrized as  $u = \{u^\alpha\}$  and the local coordinates on  $T_u \mathbf{V}^{2n}$  are labeled  $(u, z) = (u^\alpha, z^\beta)$ , where  $z^\beta$  are fiber coordinates.

The formalism of deformation quantization can be developed by using  $C^\infty(V)[[v]]$ , the space of formal series of variable  $v$  with coefficients from  $C^\infty(V)$  on a Poisson manifold  $(V, \{\cdot, \cdot\})$  (in this work, we deal with an almost Poisson structure defined by the canonical almost symplectic structure). One defines an associative algebra structure on  $C^\infty(V)[[v]]$  with a  $v$ -linear and  $v$ -adically continuous star product

$${}^1f * {}^2f = \sum_{r=0}^{\infty} {}_rC({}^1f, {}^2f) v^r, \quad (1.117)$$

where  ${}_rC, r \geq 0$ , are bilinear operators on  $C^\infty(V)$  with  ${}_0C({}^1f, {}^2f) = {}^1f {}^2f$  and  ${}_1C({}^1f, {}^2f) - {}_1C({}^2f, {}^1f) = i\{{}^1f, {}^2f\}$ ;  $i$  being the complex unity. Constructions of type (1.117) are used for stating a formal Wick product

$$a \circ b(z) \doteq \exp \left( i \frac{v}{2} \check{\mathbf{A}}^{\alpha\beta} \frac{\partial^2}{\partial z^\alpha \partial z_{[1]}^\beta} \right) a(z) b(z_{[1]}) \big|_{z=z_{[1]}}, \quad (1.118)$$

for two elements  $a$  and  $b$  defined by series of type

$$a(v, z) = \sum_{r \geq 0, |\{\alpha\}| \geq 0} a_{r, \{\alpha\}}(u) z^{\{\alpha\}} v^r, \quad (1.119)$$

where by  $\{\alpha\}$  we label a multi-index. This way, we define a formal Wick algebra  $\check{\mathbf{W}}_u$  associated with the tangent space  $T_u \mathbf{V}^{2n}$ , for  $u \in \mathbf{V}^{2n}$ . It should be noted that the fibre product (1.118) can be trivially extended to the space of  $\check{\mathbf{W}}$ -valued N-adapted differential forms  $\check{\mathcal{W}} \otimes \Lambda$  by means of the usual exterior product of the scalar forms  $\Lambda$ , where  $\check{\mathcal{W}}$  denotes the sheaf of smooth sections of  $\check{\mathbf{W}}$ . There is a standard grading on  $\Lambda$  denoted  $\deg_a$ . One also introduces gradings  $\deg_v, \deg_s, \deg_a$  on  $\mathcal{W} \otimes \Lambda$  defined on homogeneous



elements  $v, z^\alpha, \check{\mathbf{e}}^\alpha$  as follows:  $\deg_v(v) = 1$ ,  $\deg_s(z^\alpha) = 1$ ,  $\deg_a(\check{\mathbf{e}}^\alpha) = 1$ , and all other gradings of the elements  $v, z^\alpha, \check{\mathbf{e}}^\alpha$  are set to zero. In this case, the product  $\circ$  from (1.118) on  $\check{\mathcal{W}} \otimes \mathbf{\Lambda}$  is bigraded. This is written w.r.t the grading  $\text{Deg} = 2\deg_v + \deg_s$  and the grading  $\deg_a$ .

### Normal Fedosov's d-operators

The normal d-connection  $\widehat{\mathbf{D}} = \{\widehat{\Gamma}_{\alpha\beta}^\gamma\}$  (1.109) can be extended to operators

$$\widehat{\mathbf{D}}(a \otimes \lambda) \doteq \left( \check{\mathbf{e}}_\alpha(a) - u^\beta \widehat{\Gamma}_{\alpha\beta}^\gamma {}^z\check{\mathbf{e}}_\alpha(a) \right) \otimes (\check{\mathbf{e}}^\alpha \wedge \lambda) + a \otimes d\lambda, \quad (1.120)$$

on  $\check{\mathcal{W}} \otimes \mathbf{\Lambda}$ , where  ${}^z\check{\mathbf{e}}_\alpha$  is  $\check{\mathbf{e}}_\alpha$  redefined in  $z$ -variables. This operator  $\widehat{\mathbf{D}}$  is a N-adapted  $\deg_a$ -graded derivation of the distinguished algebra  $(\check{\mathcal{W}} \otimes \mathbf{\Lambda}, \circ)$ , called d-algebra. Such a property follows from (1.118) and (1.120)).

**Definition 1.2.23** *The Fedosov distinguished operators (d-operators)  $\check{\delta}$  and  $\check{\delta}^{-1}$  on  $\check{\mathcal{W}} \otimes \mathbf{\Lambda}$ , are defined*

$$\check{\delta}(a) = \check{\mathbf{e}}^\alpha \wedge {}^z\check{\mathbf{e}}_\alpha(a), \text{ and } \check{\delta}^{-1}(a) = \begin{cases} \frac{i}{p+q} z^\alpha \check{\mathbf{e}}_\alpha(a), & \text{if } p+q > 0, \\ 0, & \text{if } p=q=0, \end{cases} \quad (1.121)$$

where any  $a \in \check{\mathcal{W}} \otimes \mathbf{\Lambda}$  is homogeneous w.r.t. the grading  $\deg_s$  and  $\deg_a$  with  $\deg_s(a) = p$  and  $\deg_a(a) = q$ .

The d-operators (1.121) define the formula  $a = (\check{\delta} \check{\delta}^{-1} + \check{\delta}^{-1} \check{\delta} + \sigma)(a)$ , where  $a \mapsto \sigma(a)$  is the projection on the  $(\deg_s, \deg_a)$ -bihomogeneous part of  $a$  of degree zero,  $\deg_s(a) = \deg_a(a) = 0$ ;  $\check{\delta}$  is also a  $\deg_a$ -graded derivation of the d-algebra  $(\check{\mathcal{W}} \otimes \mathbf{\Lambda}, \circ)$ . In order to emphasize the almost Kähler structure, we used the canonical almost symplectic geometric objects defined by a fixed  $\mathcal{L}$ . Nevertheless, we can always change the "Lagrangian mechanics variables" and redefine  $\check{\theta}$ ,  $\check{\mathbf{e}}_\alpha$  and  $\widehat{\Gamma}_{\alpha\beta}^\gamma$  with respect to arbitrary frame and co-frame bases using vielbeins  $\mathbf{e}_\alpha^a$  and their duals  $\mathbf{e}_a^\alpha$ , defined by  $e^i_{\underline{i}}, e^a_{\underline{a}}$  and  $e^a_{\underline{a}}$ .

**Proposition 1.2.7** *The torsion and curvature canonical d-operators of the extension of  $\widehat{\mathbf{D}}$  to  $\check{\mathcal{W}} \otimes \mathbf{\Lambda}$ , are computed*

$${}^z\widehat{\mathcal{T}} \doteq \frac{z^\gamma}{2} \check{\theta}_{\gamma\tau} \widehat{\mathbf{T}}_{\alpha\beta}^\tau(u) \check{\mathbf{e}}^\alpha \wedge \check{\mathbf{e}}^\beta, \quad (1.122)$$

and

$${}^z\widehat{\mathcal{R}} \doteq \frac{z^\gamma z^\varphi}{4} \check{\theta}_{\gamma\tau} \widehat{\mathbf{R}}_{\varphi\alpha\beta}^\tau(u) \check{\mathbf{e}}^\alpha \wedge \check{\mathbf{e}}^\beta, \quad (1.123)$$

where the nontrivial coefficients of  $\widehat{\mathbf{T}}_{\alpha\beta}^\tau$  and  $\widehat{\mathbf{R}}_{\varphi\alpha\beta}^\tau$  are defined respectively by formulas (1.114) and (1.116).

By straightforward verifications, it follows the proof of

**Theorem 1.2.15** *The properties  $[\widehat{\mathbf{D}}, \delta] = \frac{i}{v} ad_{Wick}(z\widehat{\mathcal{T}})$  and  $\widehat{\mathbf{D}}^2 = -\frac{i}{v} ad_{Wick}(z\widehat{\mathcal{R}})$ , hold for the above operators, where  $[\cdot, \cdot]$  is the  $\deg_a$ -graded commutator of endomorphisms of  $\check{\mathcal{W}} \otimes \mathbf{\Lambda}$  and  $ad_{Wick}$  is defined via the  $\deg_a$ -graded commutator in  $(\check{\mathcal{W}} \otimes \mathbf{\Lambda}, \circ)$ .*

The above formulas can be redefined for any linear connection structure on  $\mathbf{V}^{2n}$ . For example, we consider how similar formulas can be provided for the Levi-Civita connection.

### Fedosov's d-operators and the Levi-Civita connection

For any metric structure  $\mathbf{g}$  on a manifold  $\mathbf{V}^{2n}$ , the Levi-Civita connection  $\nabla = \{ \Gamma_{\beta\gamma}^\alpha \}$  is by definition the unique linear connection that is metric compatible ( $\nabla g = 0$ ) and torsionless ( ${}_{\mathcal{T}} = 0$ ). It is not a d-connection because it does not preserve the N-connection splitting under parallel transports (1.98). Let us parametrize its coefficients in the form

$$\begin{aligned} \Gamma_{\beta\gamma}^\alpha &= ({}_1L_{jk}^i, {}_1L_{jk}^a, {}_1L_{bk}^i, {}_1L_{bk}^a, {}_1C_{jb}^i, {}_1C_{jb}^a, {}_1C_{bc}^i, {}_1C_{bc}^a), \text{ where} \\ \nabla_{\check{e}_k}(\check{e}_j) &= {}_1L_{jk}^i \check{e}_i + {}_1L_{jk}^a e_a, \quad \nabla_{\check{e}_k}(e_b) = {}_1L_{bk}^i \check{e}_i + {}_1L_{bk}^a e_a, \\ \nabla_{e_b}(\check{e}_j) &= {}_1C_{jb}^i \check{e}_i + {}_1C_{jb}^a e_a, \quad \nabla_{e_c}(e_b) = {}_1C_{bc}^i \check{e}_i + {}_1C_{bc}^a e_a. \end{aligned}$$

A straightforward calculation shows that the coefficients of the Levi-Civita connection can be expressed as

$$\begin{aligned} {}_1L_{jk}^a &= -\widehat{C}_{jb}^i \check{g}_{ik} \check{g}^{ab} - \frac{1}{2} \check{\Omega}_{jk}^a, \quad {}_1L_{bk}^i = \frac{1}{2} \check{\Omega}_{jk}^c \check{g}_{cb} \check{g}^{ji} - \Xi_{jk}^{ih} \widehat{C}_{hb}^j, \\ {}_1L_{jk}^i &= \widehat{L}_{jk}^i, \quad {}_1L_{bk}^a = \widehat{L}_{bk}^a + {}^\pm \Xi_{cd}^{ab} \circ L_{bk}^c, \quad {}_1C_{kb}^i = \widehat{C}_{kb}^i + \frac{1}{2} \check{\Omega}_{jk}^a \check{g}_{cb} \check{g}^{ji} + \Xi_{jk}^{ih} \widehat{C}_{hb}^j, \\ {}_1C_{jb}^a &= -{}^\pm \Xi_{cb}^{ad} \circ L_{dj}^c, \quad {}_1C_{bc}^a = \widehat{C}_{bc}^a, \quad {}_1C_{ab}^i = -\frac{\check{g}^{ij}}{2} \{ {}^\circ L_{aj}^c \check{g}_{cb} + {}^\circ L_{bj}^c \check{g}_{ca} \}, \end{aligned} \quad (1.124)$$

where  $e_b = \partial/\partial y^a$ ,  $\check{\Omega}_{jk}^a$  are computed as in (1.100) but for the canonical N-connection  $\check{\mathbf{N}}$  (1.94),  $\Xi_{jk}^{ih} = \frac{1}{2}(\delta_j^i \delta_k^h - \check{g}_{jk} \check{g}^{ih})$ ,  ${}^\pm \Xi_{cd}^{ab} = \frac{1}{2}(\delta_c^a \delta_d^b \pm \check{g}_{cd} \check{g}^{ab})$ ,  ${}^\circ L_{aj}^c = \widehat{L}_{aj}^c - e_a(\check{N}_j^c)$ ,  $\check{g}_{ik}$  and  $\check{g}^{ab}$  are defined for the representation of the metric in Lagrange-Finsler variables (1.92) and the normal d-connection  $\widehat{\Gamma}_{\beta\gamma}^\alpha = (\widehat{L}_{jk}^i, {}^v \widehat{C}_{bc}^a)$  (1.109) is given by coefficients (1.110).

Let introduce the distortion d-tensor  ${}_1Z^\gamma_{\alpha\beta}$  with N-adapted coefficients

$$\begin{aligned} {}_1Z^a_{jk} &= -\widehat{C}^i_{jb}\check{g}_{ik}\check{g}^{ab} - \frac{1}{2}\check{\Omega}^a_{jk}, \quad {}_1Z^i_{bk} = \frac{1}{2}\check{\Omega}^c_{jk}\check{g}_{cb}\check{g}^{ji} - \Xi^{ih}_{jk}\widehat{C}^j_{hb}, \\ {}_1Z^i_{jk} &= 0, \quad {}_1Z^a_{bk} = +\Xi^{ab}_{cd} \circ L^c_{bk}, \quad {}_1Z^i_{kb} = \frac{1}{2}\check{\Omega}^a_{jk}\check{g}_{cb}\check{g}^{ji} + \Xi^{ih}_{jk}\widehat{C}^j_{hb}, \\ {}_1Z^a_{jb} &= -\Xi^{ad}_{cb} \circ L^c_{dj}, \quad {}_1Z^a_{bc} = 0, \quad {}_1Z^i_{ab} = -\frac{g^{ij}}{2} [\circ L^c_{aj}\check{g}_{cb} + \circ L^c_{bj}\check{g}_{ca}], \end{aligned} \quad (1.125)$$

The next result follows from the above arguments.

**Proposition 1.2.8** *The N-adapted coefficients, of the normal d-connection and of the distortion d-tensors define the Levi-Civita connection as*

$${}_1\Gamma^\gamma_{\alpha\beta} = \widehat{\Gamma}^\gamma_{\alpha\beta} + {}_1Z^\gamma_{\alpha\beta}, \quad (1.126)$$

where  ${}_1Z^\gamma_{\alpha\beta}$  are given by formulas (1.126) and  $h$ - and  $v$ -components of  $\widehat{\Gamma}^\alpha_{\beta\gamma}$  are given by (1.110).

We emphasize that all components of  ${}_1\Gamma^\gamma_{\alpha\beta}$ ,  $\widehat{\Gamma}^\gamma_{\alpha\beta}$  and  ${}_1Z^\gamma_{\alpha\beta}$  are uniquely defined by the coefficients of d-metric (1.91), or (equivalently) by (1.92) and (1.94). The constructions can be obtained for any  $n + n$  splitting on  $V^{2n}$ , which for suitable  $\mathcal{L}$ , or  $\mathcal{F}$ , admit a Lagrange, or Finsler, like representation of geometric objects.

By proposition 1.2.7, the expressions for the curvature and torsion of canonical d-operators of the extension of  $\nabla$  to  $\check{W} \otimes \Lambda$ , are

$$\begin{aligned} {}^z\mathcal{R} &\doteq \frac{z^\gamma z^\varphi}{4} \check{\theta}_{\gamma\tau} {}_1R^\tau_{\varphi\alpha\beta}(u) \check{\mathbf{e}}^\alpha \wedge \check{\mathbf{e}}^\beta, \\ {}^z\mathcal{T} &\doteq \frac{z^\gamma}{2} \check{\theta}_{\gamma\tau} {}_1T^\tau_{\alpha\beta}(u) \check{\mathbf{e}}^\alpha \wedge \check{\mathbf{e}}^\beta \equiv 0, \end{aligned} \quad (1.127)$$

where  ${}_1T^\tau_{\alpha\beta} = 0$ , by definition, and  ${}_1R^\tau_{\varphi\alpha\beta}$  is computed with respect to the N-adapted Lagrange-Finsler canonical bases by introducing  $\widehat{\Gamma}^\gamma_{\alpha\beta} = -{}_1\Gamma^\gamma_{\alpha\beta} + {}_1Z^\gamma_{\alpha\beta}$ , see (1.126), into (1.116). To the N-adapted d-operator (1.120), we can associate

$$\widehat{\nabla}(a \otimes \lambda) \doteq \left( \check{\mathbf{e}}_\alpha(a) - u^\beta {}_1\Gamma^\gamma_{\alpha\beta} {}^z\check{\mathbf{e}}_\alpha(a) \right) \otimes (\check{\mathbf{e}}^\alpha \wedge \lambda) + a \otimes d\lambda, \quad (1.128)$$

on  $\check{W} \otimes \Lambda$ , where  ${}^z\check{\mathbf{e}}_\alpha$  is  $\check{\mathbf{e}}_\alpha$  redefined in  $z$ -variables. This almost symplectic connection  $\widehat{\nabla}$  is torsionless and, in general, is not adapted to the N-connection structures.

**Corollary 1.2.3** *For the Levi-Civita connection  $\nabla = \{ \Gamma_{\beta\gamma}^\alpha \}$  on a  $N$ -anholonomic manifold  $\mathbf{V}^{2n}$ , we have  $[\hat{\nabla}, \check{\delta}] = 0$  and  $\hat{\nabla}^2 = -\frac{i}{v} ad_{Wick}(\check{z}\mathcal{R})$ , where  $\hat{\nabla}$  is defined by formula (1.128),  $\check{z}\mathcal{R}$  is given by (1.127),  $[\cdot, \cdot]$  is the  $\deg_a$ -graded commutator of endomorphisms of  $\check{\mathcal{W}} \otimes \mathbf{\Lambda}$  and  $ad_{Wick}$  is defined via the  $\deg_a$ -graded commutator in  $(\check{\mathcal{W}} \otimes \mathbf{\Lambda}, \circ)$ .*

**Proof.** It is a straightforward consequence of the Theorem 1.2.15 for the Levi-Civita and curvature operators extended on  $\check{\mathcal{W}} \otimes \mathbf{\Lambda}$ .  $\square$

Prescribing a  $n + n$  splitting on  $\mathbf{V}^{2n}$ , we can work equivalently with any metric compatible linear connection structure which is  $N$ -adapted, or not, if such a connection is completely defined by the (pseudo) Riemannian metric structure. It is preferable to use the approach with the normal  $d$ -connection because this way we have both an almost symplectic analogy and Lagrange, or Finsler, like interpretation of geometric objects. In standard classical gravity, in order to solve some physical problems, it is more convenient to work with the Levi-Civita connection or its spin like representations (for instance, in the Einstein-Dirac theory). The self-dual and further generalizations to Ashtekar variables are more convenient, respectively, in canonical ADN classical and quantum gravity and/or loop quantum gravity.

It should be noted that the formulas for Fedosov's  $d$ -operators and their properties do not depend in explicit form on generating functions  $\mathcal{L}$ , or  $\mathcal{F}$ . Such a function may be formally introduced for elaborating a Lagrange mechanics, or Finsler, modeling for a (pseudo) Riemannian space with a general  $n + n$  nonholonomic splitting. This way, we emphasize that the Fedosov's approach is valid for various type of (pseudo) Riemann, Riemann-Cartan, Lagrange-Finsler, almost Kähler and other types of holonomic and non-holonomic manifolds used for geometrization of mechanical and field models. Nevertheless, the constructions are performed in a general form and the final results do not depend on any "background" structures. We conclude that  $3 + 1$  fibration approaches are more natural for loop quantum gravity, but the models with nonholonomic  $2 + 2$  splitting result in almost Kähler quantum models; although both types of quantization, loop and deformation, provide background independent constructions.

## Deformation Quantization of Einstein and Lagrange Spaces

Formulating a (pseudo) Riemannian geometry in Lagrange-Finsler variables, we can quantize the metric, frame and linear connection structures following standard methods for deformation quantization of almost Kähler

manifolds. The goal of this section is to provide the main Fedosov type results for such constructions and to show how the Einstein manifolds can be encoded into the topological structure of such quantized nonholonomic spaces.

**Fedosov's theorems for normal d-connections:** The third main result of this work will be stated below by three theorems for the normal d-connection (equivalently, canonical almost symplectic structure)  $\widehat{\mathbf{D}} \equiv {}_{\theta}\widehat{\mathbf{D}}$  (1.109).

**Theorem 1.2.16** *Any (pseudo) Riemannian metric  $\mathbf{g}$  (1.91) (equivalently,  $\mathbf{g} = \check{\mathbf{g}}$  (1.92)) defines a flat normal Fedosov d-connection  $\widehat{\mathcal{D}} := -\check{\delta} + \widehat{\mathbf{D}} - \frac{i}{v}ad_{Wick}(r)$  satisfying the condition  $\widehat{\mathcal{D}}^2 = 0$ , where the unique element  $r \in \check{\mathcal{W}} \otimes \mathbf{\Lambda}$ ,  $\deg_a(r) = 1$ ,  $\check{\delta}^{-1}r = 0$ , solves the equation  $\check{\delta}r = \widehat{\mathcal{T}} + \widehat{\mathcal{R}} + \widehat{\mathbf{D}}r - \frac{i}{v}r \circ r$  and this element can be computed recursively with respect to the total degree  $Deg$  as follows:*

$$\begin{aligned} r^{(0)} &= r^{(1)} = 0, r^{(2)} = \check{\delta}^{-1}\widehat{\mathcal{T}}, r^{(3)} = \check{\delta}^{-1} \left( \widehat{\mathcal{R}} + \widehat{\mathbf{D}}r^{(2)} - \frac{i}{v}r^{(2)} \circ r^{(2)} \right), \\ r^{(k+3)} &= \check{\delta}^{-1} \left( \widehat{\mathbf{D}}r^{(k+2)} - \frac{i}{v} \sum_{l=0}^k r^{(l+2)} \circ r^{(l+2)} \right), k \geq 1, \end{aligned}$$

where by  $a^{(k)}$  we denoted the  $Deg$ -homogeneous component of degree  $k$  of an element  $a \in \check{\mathcal{W}} \otimes \mathbf{\Lambda}$ .

**Proof.** It follows from straightforward verifications of the property  $\widehat{\mathcal{D}}^2 = 0$  using for  $r$  formal series of type (1.119) and the formulas for N-adapted coefficients: (1.110) for  $\widehat{\mathbf{D}}$ , (1.114) for  $\widehat{\mathcal{T}}$ , (1.116) for  $\widehat{\mathcal{R}}$ , and the properties of Fedosov's d-operators (1.121) stated by Theorem 1.2.15. The length of this paper does not allow us to present such a tedious calculation which is a N-adapted version for corresponding "hat" operators.  $\square$

The procedure of deformation quantization is related to the definition of a star-product which in our approach can be defined canonically because the normal d-connection  $\widehat{\mathbf{D}}$  is a N-adapted variant of the affine and almost symplectic connection considered in that work. This provides a proof for

**Theorem 1.2.17** *A star-product on the almost Kähler model of a (pseudo) Riemannian space in Lagrange-Finsler variables is defined on  $C^\infty(\mathbf{V}^{2n})[[v]]$  by formula  ${}^1f * {}^2f \doteq \sigma(\tau({}^1f)) \circ \sigma(\tau({}^2f))$ , where the projection  $\sigma : \check{\mathcal{W}}_{\widehat{\mathcal{D}}} \rightarrow$*

$C^\infty(\mathbf{V}^{2n})[[v]]$  onto the part of  $\deg_s$ -degree zero is a bijection and the inverse map  $\tau : C^\infty(\mathbf{V}^{2n})[[v]] \rightarrow \mathcal{W}_{\widehat{\mathbf{D}}}$  can be calculated recursively w.r.t the total degree  $Deg$ ,

$$\begin{aligned}\tau(f)^{(0)} &= f \text{ and, for } k \geq 0, \\ \tau(f)^{(k+1)} &= \check{\delta}^{-1} \left( \widehat{\mathbf{D}}\tau(f)^{(k)} - \frac{i}{v} \sum_{l=0}^k ad_{Wick}(r^{(l+2)})(\tau(f)^{(k-l)}) \right).\end{aligned}$$

We denote by  ${}^f\xi$  the Hamiltonian vector field corresponding to a function  $f \in C^\infty(\mathbf{V}^{2n})$  on space  $(\mathbf{V}^{2n}, \check{\theta})$  and consider the antisymmetric part  ${}^{-}C({}^1f, {}^2f) \doteq \frac{1}{2} (C({}^1f, {}^2f) - C({}^2f, {}^1f))$  of bilinear operator  $C({}^1f, {}^2f)$ . We say that a star-product (1.117) is normalized if  ${}_1C({}^1f, {}^2f) = \frac{i}{2} \{ {}^1f, {}^2f \}$ , where  $\{ \cdot, \cdot \}$  is the Poisson bracket. For the normalized  $*$ , the bilinear operator  ${}^{-}C$  defines a de Rham–Chevalley 2-cocycle, when there is a unique closed 2-form  $\check{\varkappa}$  such that  ${}_2C({}^1f, {}^2f) = \frac{1}{2} \check{\varkappa}({}^1\xi, {}^2\xi)$  for all  ${}^1f, {}^2f \in C^\infty(\mathbf{V}^{2n})$ . This is used to introduce  $c_0(*) \doteq [\check{\varkappa}]$  as the equivalence class. A straightforward computation of  ${}_2C$  and the results of Theorem 1.2.17 provide the proof of

**Lemma 1.2.4** *The unique 2-form defined by the normal d-connection can be computed as  $\check{\varkappa} = -\frac{i}{8} \check{\mathbf{J}}_\tau^{\alpha'} \widehat{\mathcal{R}}_\tau^{\alpha'} - \frac{i}{6} d \left( \check{\mathbf{J}}_\tau^{\alpha'} \widehat{\mathbf{T}}_{\alpha'\beta}^\tau \check{\mathbf{e}}^\beta \right)$ , where the coefficients of the curvature and torsion 2-forms of the normal d-connection 1-form are given respectively by formulas (1.115) and (1.113).*

We now define another canonical class  $\check{\varepsilon}$ , for  ${}^{\check{N}}T\mathbf{V}^{2n} = h\mathbf{V}^{2n} \oplus v\mathbf{V}^{2n}$ , where the left label indicates that the tangent bundle is split nonholonomically by the canonical N-connection structure  $\check{\mathbf{N}}$ . We can perform a distinguished complexification of such second order tangent bundles in the form  $T_{\mathbb{C}} \left( {}^{\check{N}}T\mathbf{V}^{2n} \right) = T_{\mathbb{C}}(h\mathbf{V}^{2n}) \oplus T_{\mathbb{C}}(v\mathbf{V}^{2n})$  and introduce  $\check{\varepsilon}$  as the first Chern class of the distributions  $T'_{\mathbb{C}} \left( {}^{\check{N}}T\mathbf{V}^{2n} \right) = T'_{\mathbb{C}}(h\mathbf{V}^{2n}) \oplus T'_{\mathbb{C}}(v\mathbf{V}^{2n})$  of couples of vectors of type  $(1, 0)$  both for the h- and v-parts. In explicit form, we can calculate  $\check{\varepsilon}$  by using the d-connection  $\widehat{\mathbf{D}}$  and the h- and v-projections  $h\Pi = \frac{1}{2}(Id_h - iJ_h)$  and  $v\Pi = \frac{1}{2}(Id_v - iJ_v)$ , where  $Id_h$  and  $Id_v$  are respective identity operators and  $J_h$  and  $J_v$  are almost complex operators, which are projection operators onto corresponding  $(1, 0)$ -subspaces. Introducing the matrix  $(h\Pi, v\Pi) \widehat{\mathcal{R}} (h\Pi, v\Pi)^T$ , where  $(\dots)^T$  means transposition, as the curvature matrix of the N-adapted restriction of of the normal d-connection  $\widehat{\mathbf{D}}$

to  $T'_\mathbb{C} \left( \check{N} T \mathbf{V}^{2n} \right)$ , we compute the closed Chern–Weyl form

$$\check{\gamma} = -iTr \left[ (h\Pi, v\Pi) \widehat{\mathcal{R}} (h\Pi, v\Pi)^T \right] = -iTr \left[ (h\Pi, v\Pi) \widehat{\mathcal{R}} \right] = -\frac{1}{4} \check{\mathbf{J}}_\tau^{\alpha'} \widehat{\mathcal{R}}^\tau_{\alpha'}. \quad (1.129)$$

We get that the canonical class is  $\check{\varepsilon} \doteq [\check{\gamma}]$ , which proves the

**Theorem 1.2.18** *The zero-degree cohomology coefficient  $c_0(*)$  for the almost Kähler model of a (pseudo) Riemannian space defined by  $d$ -tensor  $\mathbf{g}$  (1.91) (equivalently, by  $\check{\mathbf{g}}$  (1.92)) is computed  $c_0(*) = -(1/2i) \check{\varepsilon}$ .*

The coefficient  $c_0(*)$  can be similarly computed for the case when a metric of type (1.91) is a solution of the Einstein equations and this zero-degree coefficient defines certain quantum properties of the gravitational field. A more rich geometric structure should be considered if we define a value similar to  $c_0(*)$  encoding the information about Einstein manifolds deformed into corresponding quantum configurations.

### The zero-degree cohomology coefficient for Einstein manifolds

The priority of deformation quantization is that we can elaborate quantization schemes when metric, vielbein and connection fields are not obligatory subjected to satisfy certain field equations and/or derived by a variational procedure. On the other hand, in certain canonical and loop quantization models, the gravitational field equations are considered as the starting point for deriving a quantization formalism. In such cases, the Einstein equations are expressed into "lapse" and "shift" (and/or generalized Ashtekar) variables and the quantum variant of the gravitational field equations is prescribed to be in the form of Wheeler De Witt equations (or corresponding systems of constraints in complex/real generalized connection and dreibein variables). In this section, we analyze the problem of encoding the Einstein equations into a geometric formalism of deformation quantization.

**Gravitational field equations:** For any  $d$ -connection  $\mathbf{D} = \{\Gamma\}$ , we can define the Ricci tensor  $Ric(\mathbf{D}) = \{\mathbf{R}_{\beta\gamma} \doteq \mathbf{R}^\alpha_{\beta\gamma\alpha}\}$  and the scalar curvature  ${}^sR \doteq \mathbf{g}^{\alpha\beta} \mathbf{R}_{\alpha\beta}$  ( $\mathbf{g}^{\alpha\beta}$  being the inverse matrix to  $\mathbf{g}_{\alpha\beta}$  (1.91)). If a  $d$ -connection is uniquely determined by a metric in a unique metric compatible form,  $\mathbf{D}\mathbf{g} = 0$ , (in general, the torsion of  $\mathbf{D}$  is not zero, but induced canonically by the coefficients of  $\mathbf{g}$ ), we can postulate in straightforward form the field equations

$$\mathbf{R}^\alpha_\beta - \frac{1}{2}({}^sR + \lambda)\mathbf{e}^\alpha_\beta = 8\pi G \mathbf{T}^\alpha_\beta, \quad (1.130)$$

where  $\mathbf{T}_\beta^\alpha$  is the effective energy-momentum tensor,  $\lambda$  is the cosmological constant,  $G$  is the Newton constant in the units when the light velocity  $c = 1$ , and  $\mathbf{e}_\beta = \mathbf{e}_\beta^\alpha \partial / \partial u^\alpha$  is the N-elongated operator (1.3).

Let us consider the absolute antisymmetric tensor  $\epsilon_{\alpha\beta\gamma\delta}$  and effective source 3-form  $\mathcal{T}_\beta = \mathbf{T}_\beta^\alpha \epsilon_{\alpha\beta\gamma\delta} du^\beta \wedge du^\gamma \wedge du^\delta$  and express the curvature tensor  $\mathcal{R}_\gamma^\tau = \mathbf{R}_{\gamma\alpha\beta}^\tau \mathbf{e}^\alpha \wedge \mathbf{e}^\beta$  of  $\Gamma_{\beta\gamma}^\alpha = {}_1\Gamma_{\beta\gamma}^\alpha - Z_{\beta\gamma}^\alpha$  as  $\mathcal{R}_\gamma^\tau = {}_1\mathcal{R}_\gamma^\tau - \mathcal{Z}_\gamma^\tau$ , where  ${}_1\mathcal{R}_\gamma^\tau = {}_1R_{\gamma\alpha\beta}^\tau \mathbf{e}^\alpha \wedge \mathbf{e}^\beta$  is the curvature 2-form of the Levi-Civita connection  $\nabla$  and the distortion of curvature 2-form  $\mathcal{Z}_\gamma^\tau$  is defined by  $Z_{\beta\gamma}^\alpha$ . For the gravitational  $(\mathbf{e}, \Gamma)$  and matter  $\phi$  fields, we consider the effective action  $S[\mathbf{e}, \Gamma, \phi] = {}^g S[\mathbf{e}, \Gamma] + {}^{matter} S[\mathbf{e}, \Gamma, \phi]$ .

**Theorem 1.2.19** *The equations (1.130) can be represented as 3-form equations*

$$\epsilon_{\alpha\beta\gamma\tau} \left( \mathbf{e}^\alpha \wedge \mathcal{R}^{\beta\gamma} + \lambda \mathbf{e}^\alpha \wedge \mathbf{e}^\beta \wedge \mathbf{e}^\gamma \right) = 8\pi G \mathcal{T}_\tau \quad (1.131)$$

following from the action by varying the components of  $\mathbf{e}_\beta$ , when

$$\begin{aligned} \mathcal{T}_\tau &= {}^m \mathcal{T}_\tau + {}^Z \mathcal{T}_\tau, \\ {}^m \mathcal{T}_\tau &= {}^m \mathbf{T}_\tau^\alpha \epsilon_{\alpha\beta\gamma\delta} du^\beta \wedge du^\gamma \wedge du^\delta, \quad {}^Z \mathcal{T}_\tau = (8\pi G)^{-1} \mathcal{Z}_\tau^\alpha \epsilon_{\alpha\beta\gamma\delta} du^\beta \wedge du^\gamma \wedge du^\delta, \end{aligned}$$

where  ${}^m \mathbf{T}_\tau^\alpha = \delta {}^{matter} S / \delta \mathbf{e}_\tau^\alpha$  are equivalent to the usual Einstein equations for the Levi-Civita connection  $\nabla$ ,  ${}_1\mathbf{R}_\beta^\alpha - \frac{1}{2}({}_1R + \lambda)\mathbf{e}_\beta^\alpha = 8\pi G {}^m \mathbf{T}_\beta^\alpha$ .

For the Einstein gravity in Lagrange-Finsler variables, we obtain:

**Corollary 1.2.4** *The vacuum Einstein eqs with cosmological constant in terms of the canonical N-adapted vierbeins and normal d-connection are*

$$\epsilon_{\alpha\beta\gamma\tau} \left( \check{\mathbf{e}}^\alpha \wedge \widehat{\mathcal{R}}^{\beta\gamma} + \lambda \check{\mathbf{e}}^\alpha \wedge \check{\mathbf{e}}^\beta \wedge \check{\mathbf{e}}^\gamma \right) = 8\pi G {}^Z \widehat{\mathcal{T}}_\tau, \quad (1.132)$$

or, for the Levi-Civita connection,  $\epsilon_{\alpha\beta\gamma\tau} (\check{\mathbf{e}}^\alpha \wedge {}_1\mathcal{R}^{\beta\gamma} + \lambda \check{\mathbf{e}}^\alpha \wedge \check{\mathbf{e}}^\beta \wedge \check{\mathbf{e}}^\gamma) = 0$ .

**Proof.** The conditions of the mentioned Theorem 1.2.19 are redefined for the co-frames  $\check{\mathbf{e}}^\alpha$  elongated by the canonical N-connection (1.94), deformation of linear connections (1.126) and curvature (1.116) with deformation of curvature 2-form of type

$$\widehat{\mathcal{R}}_\gamma^\tau = {}_1\mathcal{R}_\gamma^\tau - \widehat{\mathcal{Z}}_\gamma^\tau. \quad (1.133)$$

We put "hat" on  ${}^Z \widehat{\mathcal{T}}_\tau$  because this value is computed using the normal d-connection.  $\square$

Using formulas (1.132) and (1.133), we can write

$$\widehat{\mathcal{R}}^{\beta\gamma} = -\lambda \check{\mathbf{e}}^\beta \wedge \check{\mathbf{e}}^\gamma - \widehat{\mathcal{Z}}^{\beta\gamma} \text{ and } {}_1\mathcal{R}^{\beta\gamma} = -\lambda \check{\mathbf{e}}^\beta \wedge \check{\mathbf{e}}^\gamma \quad (1.134)$$

which is necessary for encoding the vacuum field equations into the cohomological structure of the quantum almost Kähler model of Einstein gravity.



**The Chern–Weyl form and Einstein equations:** Introducing the formulas (1.132) and (1.134) into the conditions of Theorem 1.2.18, we obtain the forth main result in this subsection:

**Theorem 1.2.20** *The zero-degree cohomology coefficient  $c_0(*)$  for the almost Kähler model of an Einstein space defined by a d-tensor  $\mathbf{g}$  (1.91) (equivalently, by  $\check{\mathbf{g}}$  (1.92)) as a solution of (1.132) is  $c_0(*) = -(1/2i) \check{\varepsilon}$ , for  $\check{\varepsilon} \doteq [\check{\gamma}]$ , where  $\check{\gamma} = \frac{1}{4} \check{\mathbf{J}}_{\tau\alpha} \left( -\lambda \check{\mathbf{e}}^\tau \wedge \check{\mathbf{e}}^\alpha + \widehat{\mathcal{Z}}^{\tau\alpha} \right)$ .*

**Proof.** We sketch the key points of the proof which follows from (1.129) and (1.134). It should be noted that for  $\lambda = 0$  the 2-form  $\widehat{\mathcal{Z}}^{\tau\alpha}$  is defined by the deformation d-tensor from the Levi–Civita connection to the normal d-connection (1.126), see formulas (1.125). Such objects are defined by classical vacuum solutions of the Einstein equations. We conclude that  $c_0(*)$  encodes the vacuum Einstein configurations, in general, with nontrivial constants and their quantum deformations.  $\square$

If the Wheeler De Witt equations represent a quantum version of the Einstein equations for loop quantum gravity, the Chern–Weyl 2-form can be used to define the quantum version of Einstein equations (1.131) in the deformation quantization approach:

**Corollary 1.2.5** *In Lagrange–Finsler variables, the quantum field equations corresponding to Einstein’s general relativity are*

$$\check{\mathbf{e}}^\alpha \wedge \check{\gamma} = \epsilon^{\alpha\beta\gamma\tau} 2\pi G \check{\mathbf{J}}_{\beta\gamma} \widehat{\mathcal{T}}_\tau - \frac{\lambda}{4} \check{\mathbf{J}}_{\beta\gamma} \check{\mathbf{e}}^\alpha \wedge \check{\mathbf{e}}^\beta \wedge \check{\mathbf{e}}^\gamma. \quad (1.135)$$

**Proof.** Multiplying  $\check{\mathbf{e}}^\alpha \wedge$  to the above 2-form written in Lagrange–Finsler variables and taking into account (1.131), re-written also in the form adapted to the canonical N-connection, and introducing the almost complex operator  $\check{\mathbf{J}}_{\beta\gamma}$ , we get the almost symplectic form of Einstein’s equations (1.135).  $\square$

It should be noted that even in the vacuum case, when  $\lambda = 0$ , the 2-form  $\check{\gamma}$  from (1.135) is not zero but defined by  $\widehat{\mathcal{T}}_\tau = {}^Z \widehat{\mathcal{T}}_\tau$ .

Finally, we emphasize that an explicit computation of  $\check{\gamma}$  for nontrivial matter fields has yet to be performed for a deformation quantization model in which interacting gravitational and matter fields are geometrized in terms of an almost Kähler model defined for spinor and fiber bundles on spacetime. This is a subject for further investigations.

## Chapter 2

# Further Perspectives

It is possible to elaborate a quite exact research program for the next three years in relation to the fact that the applicant won recently a Romanian Government Grant IDEI, PN-II-ID-PCE-2011-3-0256, till October 2014. Section 2.1 is devoted to some important ideas and plans on future applicant's research activity using the Proposal for that Grant and other ones. In section 2.2, we speculate on possible teaching and advanced pedagogical activity.

### 2.1 Future Research Activity and Collaborations

#### 2.1.1 Scientific context and motivation

The elaboration of new geometric models and methods and their applications in physics have a number of motivations coming from modern high energy physics, gravity and geometric mechanics together with a general very promising framework to construct a "modern geometry of physics". In a more restricted context, but not less important, the geometric methods were recently applied as effective tools for generating exact solutions for fundamental physics and evolution equations.

Today, physical theories and a number of multi-disciplinary research directions are so complex that it is often very difficult to formulate, investigate and elaborate any applications without a corresponding especially closed mathematical background and inter-disciplinary approaches and methods. If in the past the physicists tried traditionally to not attack problems of "pure" mathematics, the situation has substantially changed during last 20 years. A number of mathematical ideas, notions and objects were proposed and formulated in terms of general relativity, statistics and quantum field

theory and strings. In modern fundamental physical theories, mathematics provides not only tools and methods of solution of physical problems, but governs the physicists' intuition.

The future applicant's research activity is planned in the line of the mentioned unification of mathematics and physics being stated as a present days program of developing new mathematical ideas and methods with applications in modern classical and quantum gravity, Ricci flow theory and non-commutative generalizations, geometric mechanics, stochastic processes etc. Our general research goals are related to five main directions: 1) to develop a new nonholonomic approach to geometric and deformation quantization of gravity and nonlinear systems; 2) to construct and study exact solutions in gravity and Ricci flow theory with generic local anisotropy, non-trivial topology and/or hidden noncommutative structure having motivation from string/brane and extra dimension gravity theories; 3) to elaborate a corresponding formalism of nonholonomic Dirac operators and generalizations to noncommutative geometry and evolution models of fundamental geometric objects, exact solutions and quantum deformations; 4) modified theories of gravity, exact solutions and quantization methods; 5) anisotropic and non-holonomic configurations in modern cosmology and astrophysics related to dark energy/matter problems.

We shall focus on new features of the geometry of nonlinear connections and nonholonomic and quantum deformations and study new aspects related to solitonic hierarchies, bi-Hamilton formalism, (non) commutative almost symplectic structures and analogous models of Lagrange-Finsler and Hamilton-Cartan geometries, differential geometry of superspaces, fractional derivatives and dimensions, stochastic anisotropic processes etc.

### 2.1.2 Objectives

There are formulated five main objectives with respective exploratory importance, novelty, interdisciplinary character and possible applications:

1. Objective 1. Geometric and Deformation Quantization of Gravity and Matter Field Interactions and Nonholonomic Mechanical Systems.
  - In a general geometric approach, theories of classical and quantum interactions with gravitational field equations for the Levi-Civita connection can be re-formulated equivalently in almost Kähler and/or Lagrange-Finsler variables on nonholonomic manifolds and bundle spaces. Such nonholonomic configurations and/

or dynamical systems are determined by corresponding non-integrable distributions on space/-time manifolds. Our goal and novelty are oriented to geometric quantization and renormalization schemes for nonlinear theories following the nonlinear connection formalism and techniques originally elaborated for quantum Kähler and almost symplectic geometries.

- We shall compare the new approach to geometric quantization of nonholonomic almost Kähler spaces with former our constructions performed for deformation and A-brane quantization. We shall analyze possible connections and find key differences between geometric schemes and physically important perturbative models with renormalization of analogous/emergent commutative and noncommutative gravitational and gauge like theories.
- There will be provided a series of applications of geometric and deformation quantization methods to theories with nonlinear dispersions, local anisotropy and Lorenz violation induced from quantum gravity and/or string/brane theories. Such models can be described as analogous classical/quantum Lagrange-Hamilton and Finsler-Cartan geometries which will extend the research with applications in modern mechanics and nonlinear dynamics.

## 2. Objective 2. Exact Solutions in Gravity and Ricci Flow Theory

- Via nonholonomic (equivalently, anholonomic) deformations of the frame and connection structures, the Einstein field equations can be decoupled and solved in very general forms. This allows us to generate various classes of off-diagonal solutions with associated solitonic hierarchies, characterized by stochastic, fractional, fractal behavior and nonholonomic dynamical multipole moments.
- Following new geometric techniques, we shall derive new classes of locally anisotropic black holes, ellipsoids, wormholes and cosmological spacetimes. This requests new ideas and methods and generalizations for black hole uniqueness and non-hair theorems in general relativity and modified gravity theories. Various examples of vacuum and non-vacuum metrics with noncommutative, supersymmetric and/or nonsymmetric variables will be constructed and analyzed in explicit form.
- Nonholonomically constrained Ricci flows of (semi) Riemannian geometries result, in general, in various classes of commutative

and noncommutative geometries. A special interest presents the research related to geometric evolution of Einstein metrics and possible connections to beta functions and renormalization in quantum gravity. Encoding geometric and physical data in terms of almost Kähler geometry, the classical and quantum evolution scenarios can be performed and studied following our former approach elaborated for commutative and noncommutative evolution of Einstein and/or Finsler geometries.

### 3. Objective 3. Nonholonomic Clifford Structures and Dirac Operators

- The geometry of nonholonomic Clifford and spinor bundles enabled with nonlinear connections was elaborated in our works following methods of classical and quantum Lagrange–Finsler geometry. It is important to extend such constructions to nonholonomic spinor and twistor structures derived via almost Kähler and/or nonholonomic variables for certain important models of classical and quantum gravity. The concept of nonholonomic Dirac operators for almost symplectic classical and quantum systems will be developed in connection to new spinor and twistor methods in gravity and gauge models and exact solutions for Einstein–Yang–Mills–Dirac systems.
- Quantization of nonholonomic Clifford and related almost Kähler structures will be performed following geometric and deformation quantization techniques. Possible connections to twistor diagrams formalism and quantization will be analyzed.
- The theory of nonholonomic Dirac operators and spectral triples and functionals consists a fundamental mathematical background for various approaches to noncommutative geometry, particle physics and Ricci flow evolution models. Our new idea is to study nonholonomic and noncommutative Clifford structures and their geometric evolution scenarios using almost Kähler and spinor variables. A comparative study with noncommutative models derived via Seiberg–Witten transforms and deformation quantization will be performed.

### 4. Objective 4. Modified theories of gravity, exact solutions and quantization methods

- We shall extend our methods of constructing generic off–diagonal

exact solutions in Einstein gravity and (non) commutative Einstein–Finsler gravity theories to modified theories of gravity with anisotropic scaling and nonlinear dependence on scalar curvature, torsion components, variation of constants etc. The conditions when such effective theories can be modeled by nonholonomic constraints and nonlinear off–diagonal interactions will be formulated in a geometric form. There will be elaborated analytic and computer modeling programs for solitonic interactions, black hole configurations, and other classes of exact solutions. The criteria when analogous Dirac operators for almost symplectic classical and quantum systems can be considered and exact solutions for Einstein–Yang–Mills–Dirac systems in modified gravity will be constructed.

- There are known perturbative theories of gravity, in covariant and generalize non–perturbative forms, which seem to provide physically important scenarios for quantum gravity, Lorentz violations, super-luminal effects etc. We shall be interested to develop certain geometric schemes for quantization of nonholonomic Clifford and related almost Kähler structures in modified gravity and theories with off–diagonal analogous configurations.

5. Objective 5. Anisotropic and nonholonomic configurations in modern cosmology and astrophysics related to dark energy/matter problems

- This direction is strongly related to "changing of paradigms" in modern fundamental physics. In some sense, it depends on experimental and observational data and phenomenology in gravity and particle physics. We suppose to apply our experience on geometric methods and mathematical physics extended to computer modeling and graphics.
- Possible tests and phenomenology for (non) commutative and quantum gravity models, with almost Kähler and spinor variables, generalize Seiberg–Witten transforms and deformation quantization, will be proposed and analyzed in details.

### 2.1.3 Methods and approaches

#### Techniques, Milestones and Objectives:

The research methods span pure geometry, partial differential equations and analytic methods of constructing of exact solutions and important issues

in mathematical physics and theoretical particle physics in equal measure. Certain problems of geometric mechanics, diffusion theory, fractional differential geometry related to nontrivial solutions in gravity and quantization will be also concerned. Although the bulk motivation of the tasks comes from fundamental gravity and particle physics, the approach to the Project objectives is a patient and systematic development of what one believes to be the necessary and inevitable application of geometrical tools from almost Kähler geometry, generalized Finsler geometry and nonholonomic manifolds. In the process, it is planned to contribute significantly to geometric and deformation quantization and renormalization of nonholonomic Einstein and generalized Lagrange–Finsler gravity models elaborated on Lorenz space-times and, respectively, on (co) tangent bundles to such manifolds.

We shall also consider both rigorous mathematical issues on uniqueness of solutions, the simplest examples of exact solutions of fundamental field and evolution equations and their encoding as bi-Hamilton structures and solitonic hierarchies. There will be investigated the fundamental relations between nonholonomic structures and quantum geometries and duality and deformation quantization of almost Kähler models of nonholonomic pseudo-Riemann. As phase space constructions they will provide a good challenge for noncommutative Lagrange and Hamilton geometry. The methodology will consist broadly in looking at such almost symplectic geometry methods applied both on (semi) Riemannian and Riemann–Cartan manifolds and (co) tangent bundle in order to elaborate a Fedosov type formalism related to Lagrange and Hamilton geometries. We shall combine the approach with our previous results and methods on generalized Finsler (super) geometries and the anholonomic frame method in various models of gravity and strings. Let us state the specific particularities with respect to assigned number of Objectives (Obj.) in previous section:

For Obj. 1 oriented to geometric quantization of nonlinear physical systems, gravity and matter fields and mechanics: Our approach is supposed to be a synthesis of quantization schemes with nonholonomic distributions elaborated in our recent papers oriented applications to quantum gravity, geometric mechanics and almost symplectic geometries. There will be involved new issues connected to geometric quantization of almost Kähler geometries and quantization of Einstein and Einstein–Finsler spaces. As intermediate milestones there will be considered and developed certain explicit computations for perturbative models, diagrammatic techniques and nonholonomic geometric renormalization of analogous/emergent commutative and noncommutative gravitational and gauge like theories. In explicit form, we shall compute observable effects with nonlinear dispersions, local

anisotropy and Lorenz violation determined from quantum gravity models on pseudo-Riemannian spacetimes and their tangent bundle extensions. Possible effective corrections for quantum Lagrange-Hamilton spaces, quantum solitons and solitonic hierarchies will be computed using analytic methods.

For Obj. 2 on exact solutions in gravity and Ricci flow theory: Explicit study of solutions for evolution equations and systems of nonlinear partial differential equations (NPDE) modeling field interactions subjected to nonholonomic constraints positively impose a coordinate / index style for geometric and functional analysis constructions. Such coordinate-type formalism, tensor-index formulas and corresponding denotations are typical in Hamilton's and Grisha Perelman's works. Additional geometric studies are necessary to state constructions in global form, for instance, with the aim to derive global symmetries and study of certain nontrivial topological configurations. We shall elaborate a distinguished tensor calculus, with respect to frames adapted to the nonlinear connection and generalized nonholonomic structures in order to be able to compute the evolution of such objects under Ricci flows. As a first intermediate milestone step we shall elaborate coordinate free criteria stating the conditions when nonholonomic deformations of the frame and connection structures result in decoupling of the Einstein field equations and formulating of general solutions in abstract/global forms. We shall use for such constructions our former results on associated solitonic hierarchies and multipole moment formalism. The next step will be oriented to geometric methods and generalizations of sigma models and nonholonomic deformation methods for generalized black hole uniqueness and non-hair theorems in general relativity and modified gravity theories. The third step (intermediate milestone) will be related to computations for certain types of commutative and noncommutative and/or nonholonomic variables. The geometry of hypersurfaces and possible relations to global solutions of evolution of exact solutions of Einstein equations, possible connections to beta functions and renormalization in quantum gravity will be applied for classification purposes and study of possible implications in modern cosmology and astrophysics.

For Obj. 3-5: We shall develop and apply an abstract index spinor techniques adapted to nonlinear connections in Clifford bundles. The theory of nonholonomic Dirac operators will be formulated in a form admitting straightforward extensions to complex manifolds, almost symplectic spinors, nonholonomic spinor spaces and noncommutative generalizations. Methods of twistor geometry and applications to self-dual Einstein and Yang-Mills systems will be generalized for nonholonomic configurations and Pfaff systems associated to twistor equations. The approach with twistor diagrams and



quantization will be extended to nonholonomic gravitational gauge interactions and applied to almost symplectic manifolds and bundle spaces. A recent techniques of distinguished spectral triples and nonlomic Dirac operators will be applied for generating nonholonomic and/or noncommutative Clifford structures. We shall compute generalized series decompositions for Seiberg-Witten transforms of gauge like reformulated Einstein equations, deformation quantization of noncommutative spacetimes and their geometric evolution.

Finally, it is noted that there not presented comments in explicit form on techniques and milestones for Objs. 4 and 5 because such issues are on constant modification depending on observational and experimental data. There are planned some important International Conferences on Gravity and Cosmology for the second part of 2012 which will allow to formulate more exact plans on activity in such directions.

### **Travels and human and material resources**

The applicant have a more than 15 year experience as a senior researcher (CS1) and administrative charge as an expert in the fields of mathematical and theoretical physics, geometry and physics, evolution equations in physics, deformation quantization, quantization, analogous gravity, application in cosmology and astrophysics etc.

He is the researcher planned to have most travels with lectures and talks at conferences in Western Countries and Romania, all related to the Project IDEI and other funds. He is assisted by a technician (employed for 36 months) with specific skills on performing mathematical works, schemes, posters, latex and beamer arrangements of manuscripts, posters, slides etc.

For a successful realization of this multi-disciplinary research program (based on geometric methods and new directions in modern mathematics, computer methods etc), he has a very important professional support from members of traditionally strong school of differential geometry and applications existing at the Department of Mathematics of UAIC, Institute of Mathematics at Iași etc. It is planned to involve post-graduate students in the project at least with a half charge during 36 months (depending on financial sources). It is also supposed that the applicant as a leader of project may invite some researchers outside Romania for a period up till 3 months.

Finally, in this section, it should be emphasized that the Project IDEI allows the applicant to get 3 very desk tops/ laptops and computer macros necessary for advanced research on mathematics, physics, astronomy and mechanics.

## 2.2 Supervision and Pedagogical Activity

Applicant's activity after obtaining PhD in 1994 is a typical research one (beginning 1996, as a senior researcher) for mathematical physics scientists originating from former URSS and with "high mobility" in Western Countries determined by a number of research grants and fellowships and certain human rights issues. Nevertheless, it has a pluralistic pedagogical activity and experience which can be quite important for his possible future senior research positions in Romania.

### 2.2.1 Teaching and supervision experience

In brief, one should be mentioned such activities:

1. *University teaching in English, Romanian and Russian:* During 1996-1997, 2001, 2006, the applicant delivered lectures with seminars and labs activities respectively at two Universities in R. Moldova (Free University of Moldova and Academy of Economic Studies at Chişinău; it was an attempt to introduce teaching in English for students at some departments), California State University at Fresno, USA, and Brock University, Ontario Canada in such directions: a) higher mathematics, mathematical programming, statistics and probability for economists; b) physics lab; c) partial differential equations and discrete optimization. A typical course of lectures, with problems and computer lab elaborated by the applicant can be found in the Web, see [73].
2. In 2000, the applicant supervised a *Republican Seminar on "Geometric Methods in String Theory and Gravity"* at the Institute of Applied Physics, Academy of Sciences, R. Moldova. He and the bulk of that participants moved their activities as (post-graduate) students and researchers in Western Countries, after 2001. Let us consider some explicit examples of common research, publications in high influence score journals, local journals and International Conferences: a) papers on twistors and conservation laws in modified gravity, in collaboration with S. Ostaf during 1993-1996, [82, 66]; research on gauge like Finsler-Lagrange gravity, together with Yu. Goncharenko (1995), [11]; research and a series of publications and collaboration with E. Gaburov and D. Gonța (2000-2001), and with Prof. P. Stavrinou (Athens, Greece), see a series of works in monograph [70], on metric-affine Lagrange-Finsler gravity, exact solutions in such theories and brane physics; a series of important papers was published

together with Nadejda A. Vicol [one paper together with I. Chiosa, and other two students from Chisinau, and Prof. D. Singleton, USA, and Profs. P. Stavrinou and G. Tsagas, Greece], see [33, 83, 87, 89, 19], on spinors in generalized Finsler–Lagrange (super) spaces, wormholes, noncommutative geometry etc.

3. During his fellowships and visits in Europe and North America, the applicant collaborated and published papers with young researchers (master students, post-graduates and post-docs): from Spain (2004–2007), J. F. Gonzalez–Hernandes [36] and R. Santamaria (also with Prof. F. Etayo) [29]; from Romania (2001–2002), F. C. Popa [20, 66] and O. Tintareanu–Mircea [22, 65], (post-graduates of Prof. M. Visinescu), on locally anisotropic Taub NUT spinning spaces, Einstein–Dirac solitonic waves, locally anisotropic superspaces etc.
4. In R. Moldova, the applicant had the right to supervise PhD, master and diploma theses, with different competencies, during 1993–2001.

### 2.2.2 Future plans

The Habilitation Thesis would allow the applicant to supervise PhD thesis in Romania. He may use his former experience (more than 5 years of pluralistic activity) in such directions:

1. involve young researchers in activities related to his grant IDEI etc
2. organize a seminar (similarly to point 2 in previous subsection) on "Geometric Methods in Modern Physics" at UAIC and other universities and research centers
3. perform PhD supervision and research collaborations with post-docs, students etc from various countries
4. organize advance teaching on math and physics in Romanian and English, with possible visits and collaborations with researches from various places

It should be concluded that main activity of the applicant, for the future, is supposed to be a senior research one (a Romanian equivalent to Western "research professor") with certain additional advanced pedagogical activity.

## Chapter 3

# Publications, Conferences and Talks

In this Chapter, there are listed a series of "most important" applicant's publications and last 7 years conference/seminar activity (see additional information in his complete Publication List included in the file for Habilitation Thesis<sup>1</sup>). In brief, the Bibliography is presented in this form:

- ten selected most important papers, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10];
- important ISI and high influence absolute score papers relevant to the first ones are with numbers [11]-[64];
- works published in Romania, [65]-[69];
- books and reviews of books and encyclopedia are with numbers [70]-[75], chapters and sections in books and collections are [81]-[83];
- publications in R. Moldova, [76]-[80];
- papers published in proceedings of conferences, [84]-[91];
- communications and participation at conferences and seminars with support of organizers/hosts, [92]-[131];
- two electronic preprints [132, 133] (from more than 100 ones) are listed because they contain some important references and computations.

Additional references and citation of works by other authors can be found in the mentioned works.

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<sup>1</sup>see also applicant's papers in <http://inspirehep.net> and/ or [arXiv.org](http://arXiv.org)

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